Section 5.2 **The definite integral**

Definition of a definite integral.
If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition points x_0, x_1,...,x_n, where

$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

Choose points $x_i^* \in [x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral of f from a to b** is

$$\boxed{\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i}$$

if this limit exists. If the limit does exist, then f is called **integrable** on the interval $[a, b]$.

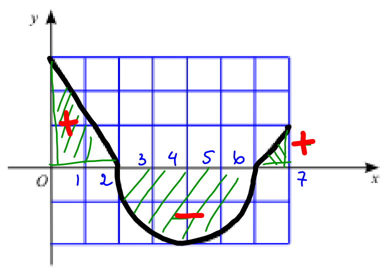
In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the limits of integration; a is the **lower limit** and b is the **upper limit**.
The procedure of calculating an integral is called **integration**.

For the special case where $f(x) \geq 0$, $\int_a^b f(x)dx$ = area under the graph of f from a to b.

$$\boxed{\int_a^b f(x)dx = -\int_b^a f(x)dx}$$

$$\boxed{\int_a^a f(x)dx = 0}$$

Example 1. Evaluate $\int_0^7 f(x)dx$ if the graph of the function $f(x)$ is



$$\int_0^7 f(x)dx = 3 \cdot 2 \cdot \frac{1}{2} - \frac{\pi(2^2)}{2} + \frac{1}{2}(1)(1)$$

$$= \boxed{3 - 2\pi + \frac{1}{2}}$$

Theorem 1. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

If f has a finite number of discontinuities and these are all jump discontinuities, then f is called **piecewise continuous function**.

Theorem 2. If f is piecewise continuous on $[a,b]$, then f is integrable on $[a,b]$.

f is integrable on $[a,b]$, then f must be **bounded function** on $[a,b]$: that is, there exist a number M such that $|f(x)| \leq M$ for all $x \in [a,b]$.

Let P be a regular partition of $[a,b]$: that is $\Delta x = \Delta x_1 = \Delta x_2 = \ldots = \Delta x_n = \frac{b-a}{n}$ and $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$,...,$x_n = b$
If we choose x_i^* to be the **right endpoint** of the ith interval, then $x_i^* = x_i = a + i\Delta x = a + i\frac{b-a}{n}$, so

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)}$$

If x_i^* is the **midpoint** of the interval ith interval, then $x_i^* = \bar{x}_i = (x_{i-1} + x_i)/2$, so

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(\bar{x}_i)}$$

Example 2. Evaluate the integral $\int_1^4 (x^2 - 2)dx$

Properties of the definite integral

1. $\int_a^b c\,dx = c(b-a)$, where c is a constant.

2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is a constant.

3. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

4. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$.

6. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

7. If $f(x) \geq 0$ for $a < x < b$, then $\int_a^b f(x)dx \geq 0$.

8. If $f(x) \geq g(x)$ for $a < x < b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

9. If $m \leq f(x) \leq M$ for $a < x < b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

10. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Example 3. Express the limit as a definite integral

(a.) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5} = \lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\frac{1}{n}}_{\Delta x}\underbrace{\frac{i^4}{n^4}}_{f(x_i^*)} = \int_a^b f(x)dx = \boxed{\int_0^1 x^4 dx}$

$f(x_i^*) = \left(\frac{i}{n}\right)^4 \Rightarrow f(x) = x^4$

$x_i^* = \frac{i}{n}$ $\quad x_0 = \frac{0}{n} = a$, $\quad b = x_n = \frac{n}{n}$

(b.) $\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\left[3\left(1+\frac{2i}{n}\right)^5 - 6\right]}_{f(x_i^*)}\underbrace{\frac{2}{n}}_{\Delta x} = \boxed{\int_1^3 (3x^5-6)dx}$

$f(x_i^*) = 3\left(1+\frac{2i}{n}\right)^5 - 6$, $\quad 1+\frac{2i}{n} = x_i^*$

3

$x_i^* = 1 + \frac{2i}{n}$, $\quad f(x_i^*) = 3(x_i^*)^5 - 6$

$f(x) = 3x^5 - 6$

$a = x_0 = 1 + \frac{2\cdot 0}{n} = 1$

$b = x_n = 1 + \frac{2\cdot n}{n} = 3$

Example 4. Write the given sum or difference as a single integral

(a.) $\underbrace{\int_1^3 f(x)dx + \int_3^6 f(x)dx} + \int_6^1 2f(x)dx$

$$\int_1^6 f(x)dx - \int_1^6 2f(x)dx = -\int_1^6 f(x)dx$$

(b.) $\int_2^{10} f(x)dx - \int_2^7 f(x)dx = \cancel{\int_2^7 f(x)dx} + \int_7^{10} f(x)dx - \cancel{\int_2^7 f(x)dx} = \int_7^{10} f(x)dx$