Section 5.3 **The fundamental theorem of calculus.**

Theorem. Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$. [$g(x)$ is an antiderivative for $f(x)$].
2. $\boxed{\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)}$, where F is an antiderivative of f.

Example 1. Find the derivative of the function.

1. $g(x) = \int_\pi^x \frac{1}{1+t^4}dt$

$$\boxed{g'(x) = \frac{1}{1+x^4}}$$

2. $f(x) = \int_x^4 (2+\sqrt{t})^8 dt = -\int_4^x (2+\sqrt{t})^8 dt$

$$f'(x) = -(2+\sqrt{x})^8$$

---

3. $y = \int_{\tan x}^{17} \sin(t^4)dt = -\int_{17}^{\tan x} \sin(t^4)\, dt \quad \Big| u = \tan x \Big|$

$$y' = -\sin [u(x)]^4 \cdot u'(x)$$

$$= \boxed{-\sec^2 x \cdot \sin(\tan^4 x)}$$

Example 2. Evaluate the integral.

$\int y^n dy = \frac{y^{n+1}}{n+1} + C$, $n \neq -1$

$y^{1/2} \cdot y^{-2} = y^{1/2-2} = y^{-3/2}$

1. $\int_2^6 \frac{1+\sqrt{y}}{y^2} dy = \int_2^6 (1+\sqrt{y}) y^{-2} dy = \int_2^6 (y^{-2} + \sqrt{y} \cdot y^{-2}) dy$

$= \int_2^6 (y^{-2} + y^{-3/2}) dy = \left[\frac{y^{-2+1}}{-2+1} + \frac{y^{-3/2+1}}{-3/2+1} \right]_2^6 = \left(-y^{-1} - 2y^{-1/2} \right)_2^6$

$= \left(-\frac{1}{6} - 2(6)^{-1/2}\right) - \left(-\frac{1}{2} - 2\cdot(2)^{-1/2}\right) = \boxed{-\frac{1}{6} - \frac{2}{\sqrt{6}} + \frac{1}{2} + \frac{2}{\sqrt{2}}}$

2. $\int_0^4 (4-t)\sqrt{t}\, dt = \int_0^4 (4\sqrt{t} - t\sqrt{t}) dt = \int_0^4 (4t^{1/2} - t^{3/2}) dt$

$= \left[4 \frac{t^{1/2+1}}{1/2+1} - \frac{t^{3/2+1}}{3/2+1} \right]_0^4 = \left[4 \cdot \frac{2}{3} t^{3/2} - \frac{2}{5} \cdot t^{5/2} \right]_0^4$

$= \frac{8}{3} 4^{3/2} - \frac{2}{5} 4^{5/2} - 0 = \boxed{\frac{8}{3} \cdot 8 - \frac{2}{5} \cdot 32}$

3. $\int_0^3 (2\sin x - e^x)\, dx = \left[-2\cos x - e^x \right]_0^3 = -2\cos 3 - e^3 - (-2\cancel{\cos 0}^{1} - \cancel{e^0}^{1})$

$= \boxed{-2\cos 3 - e^3 + 3}$

$\int a^x dx = \frac{a^x}{\ln a} + C$

4. $\int_0^4 2^s\, ds = \left. \frac{2^s}{\ln 2} \right]_0^4 = \frac{2^4}{\ln 2} - \frac{2^0}{\ln 2} = \boxed{\frac{15}{\ln 2}}$

$\int \frac{dx}{1+x^2} = \arctan x + C$

5. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx = 8 \arctan x \Big|_{1/\sqrt{3}}^{\sqrt{3}} = 8\left(\cancel{\arctan \sqrt{3}}^{\pi/3} - \cancel{\arctan \frac{1}{\sqrt{3}}}^{\pi/6}\right) = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

$= \boxed{\frac{8\pi}{6}} = \boxed{\frac{4\pi}{3}}$

6. $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx$ $= 4\arcsin x \Big|_{1/2}^{1/\sqrt{2}} = 4\left[\arcsin\frac{1}{\sqrt{2}} - \arcsin\frac{1}{2}\right]$

$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C}$

$= 4\left[\frac{\pi}{4} - \frac{\pi}{6}\right] = 4\,\frac{3\pi - 2\pi}{12} = \frac{4\pi}{12} = \boxed{\frac{\pi}{3}}$

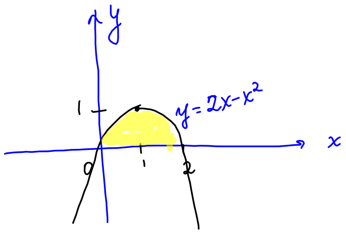
7. $\displaystyle\int_0^2 f(x)dx$, where $f(x) = \begin{cases} x^4 & 0 \le x < 1 \\ x^5 & 1 \le x \le 2 \end{cases}$

$= \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 x^4\,dx + \int_1^2 x^5\,dx$

$= \frac{x^5}{5}\Big|_0^1 - \frac{x^6}{6}\Big|_1^2 = \frac{1}{5} - 0 - \frac{2^6}{6} + \frac{1}{6}$

$= \boxed{\frac{1}{5} - \frac{63}{6}}$

Example 3. Find the area of the region enclosed by the parabola $y = 2x - x^2$ and the line $y = 0$.



$2x - x^2 = 0$

$x(2-x) = 0, \quad x_1 = 0, \quad x_2 = 2.,$ $\boxed{0 \le x \le 2}$

$$A = \int_0^2 (2x - x^2)\,dx$$

$$= \left(\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right)\Big|_0^2 = 2^2 - 0 - \frac{2^3}{3} + 0 = 4 - \frac{8}{3} = \boxed{\frac{4}{3}}$$