Section 5.4 **Indefinite integrals and the Net Change Theorem.**

Indefinite Integrals.

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals.

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation between antiderivatives and integrals given by the Fundamental Theorem, the notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an indefinite integral. Thus

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

So we can regard an indefinite integral as representing an entire family of functions (one antiderivative for each value of the constant).

NOTE: $\int_a^b f(x)dx$ is a **number** while $\int f(x)dx$ is a **function**.

Table of indefinite integrals

a) $\int a dx = ax + C,\ a$ is a constant

b) $\int x dx = \frac{x^2}{2} + C$

c) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

d) $\int \frac{dx}{x} = \ln|x| + C$

e) $\int e^x dx = e^x + C$

f) $\int a^x dx = \frac{a^x}{\ln a} + C$

g) $\int \sin x dx = -\cos x + C$

h) $\int \cos x dx = \sin x + C$

i) $\int \tan x dx = -\ln|\cos x| + C = \ln|\sec x| + C$

j) $\int \cot x dx = \ln|\sin x| + C = -\ln|\csc x| + C$

k) $\int \sec^2 x dx = \tan x + C$

l) $\int \csc^2 x dx = -\cot x + C$

m) $\int \sec x \tan x dx = \sec x + C$

n) $\int \csc x \cot x dx = -\csc x + C$

o) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$

p) $\int \frac{dx}{\sqrt{x^2 + a}} = \ln|x + \sqrt{x^2 + a}| + C$

q) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

r) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

Example 1. Evaluate the integral.

1. $\int (x^{1.3} + 7x^{2.6})dx$

2. $\displaystyle\int \left(\frac{1+r}{r}\right)^2 dr$

3. $\displaystyle\int \sec t(\sec t + \tan t)dt$

Net Change Theorem.

Part 2 of the Fundamental Theorem says that if f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b . [Note that could, for instance, increase, then decrease, then increase again. Although y might change in both directions, $F(b) - F(a)$ represents the **net** change in y.] So we can reformulate FTC2 in words as follows.

Net Change Theorem The integral of the rate change is the net change

$$\boxed{\int_a^b F'(x)dx = F(b) - F(a)}$$

This principle can be applied to all of the rates of change in the natural and social sciences. Here are a few instances of this idea:

- If V is the volume of water in a reservoir at time t , then its derivative V' is the rate at which water flows into the reservoir at time t . So

$$\int_{t_1}^{t_2} V'(t)dt = V(t_1) - V(t_2)$$

  is the change in the amount of water in the reservoir between time t_1 and time t_2 .

- If C is the concentration of the product of a chemical reaction at time T, then the rate of reaction is the derivative C'. So

$$\int_{t_1}^{t_2} C'(t)dt = C(t_1) - C(t_2)$$

is the change in the concentration of from time t_1 to time t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_{a}^{b} \rho(x)dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

- If the rate of growth of a population is $\dfrac{dP}{dT}$, then

$$\int_{t_1}^{t_2} P'(t)dt = P(t_1) - P(t_2)$$

is the net change in population during the time period from t_1 to t_2. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\textbf{displacement} = \int_{t_1}^{t_2} v(t)dt = s(t_1) - s(t_2)$$

is the net change of position, or displacement, of the particle during the time period from t_1 to t_2.

- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

$$\textbf{distance} = \int_{t_1}^{t_2} |v(t)|dt$$

Example 2. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$.

1. Find the displacement of the particle during the time period $1 \leq t \leq 6$.

2. Find the distance traveled during this time period.

Example 3. The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kg/m, where x is measured in meters from one end of the rod. Fina the total mass of the rod.

Example 4. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.