

Section 5.4 Indefinite integrals and the Net Change Theorem.

**Indefinite Integrals.**

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals.

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation between antiderivatives and integrals given by the Fundamental Theorem, the notation  $\int f(x)dx$  is traditionally used for an antiderivative of  $f$  and is called an indefinite integral. Thus

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

So we can regard an indefinite integral as representing an entire family of functions (one antiderivative for each value of the constant ).

**NOTE:**  $\int_a^b f(x)dx$  is a **number** while  $\int f(x)dx$  is a **function**.

**Table of indefinite integrals**

a)  $\int adx = ax + C$ ,  $a$  is a constant

b)  $\int xdx = \frac{x^2}{2} + C$

c)  $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ ,  $n \neq -1$

d)  $\int \frac{dx}{x} = \ln|x| + C$

e)  $\int e^x dx = e^x + C$

f)  $\int a^x dx = \frac{a^x}{\ln a} + C$ ,  $a > 0$ ,  $a \neq 1$

g)  $\int \sin x dx = -\cos x + C$

h)  $\int \cos x dx = \sin x + C$

i)  $\int \tan x dx = -\ln|\cos x| + C = \ln|\sec x| + C$

j)  $\int \cot x dx = \ln|\sin x| + C = -\ln|\csc x| + C$

k)  $\int \sec^2 x dx = \tan x + C$

l)  $\int \csc^2 x dx = -\cot x + C$

m)  $\int \sec x \tan x dx = \sec x + C$

n)  $\int \csc x \cot x dx = -\csc x + C$

o)  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$

p)  $\int \frac{dx}{\sqrt{x^2 + a}} = \ln|x + \sqrt{x^2 + a}| + C$

q)  $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

r)  $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$

**Example 1.** Evaluate the integral.

$$\begin{aligned} 1. \int (x^{1.3} + 7x^{2.6}) dx &= \frac{x^{1.3+1}}{1.3+1} + 7 \cdot \frac{x^{2.6+1}}{2.6+1} + C \\ &= \boxed{\frac{x^{2.3}}{2.3} + 7 \cdot \frac{x^{3.6}}{3.6} + C} \end{aligned}$$

1

$$\begin{aligned} 2. \int \left(\frac{1+r}{r}\right)^2 dr &= \int \frac{(1+r)^2}{r^2} dr = \int (1+r)^2 r^{-2} dr = \int \overbrace{(1+2r+r^2)}^{(1+r)^2} r^{-2} dr \\ &= \int (r^{-2} + 2r^{-1} + 1) dr = \frac{r^{-2+1}}{-2+1} + 2 \ln|r| + r + C = \boxed{-\frac{1}{r} + 2 \ln|r| + r + C} \end{aligned}$$

$$\begin{aligned} 3. \int \sec t (\sec t + \tan t) dt &= \int [\sec^2 t + \sec t \tan t] dt = \int \sec^2 t dt + \int \sec t \tan t dt \\ &= \boxed{\tan t + \sec t + C} \end{aligned}$$

**Net Change Theorem.**

Part 2 of the Fundamental Theorem says that if  $f$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where  $F$  is any antiderivative of  $f$ . This means that  $F' = f$ , so the equation can be rewritten as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

We know that  $F'(x)$  represents the rate of change of  $y = F(x)$  with respect to  $x$  and  $F(b) - F(a)$  is the change in  $y$  when  $x$  changes from  $a$  to  $b$ . [Note that could, for instance, increase, then decrease, then increase again. Although  $y$  might change in both directions,  $F(b) - F(a)$  represents the **net** change in  $y$ .] So we can reformulate FTC2 in words as follows.

**Net Change Theorem** The integral of the rate change is the net change

$$\int_a^b F'(x)dx = F(b) - F(a)$$

---

This principle can be applied to all of the rates of change in the natural and social sciences. Here are a few instances of this idea:

- If  $V$  is the volume of water in a reservoir at time  $t$ , then its derivative  $V'$  is the rate at which water flows into the reservoir at time  $t$ . So

$$\int_{t_1}^{t_2} V'(t) dt = V(t_1) - V(t_2)$$

is the change in the amount of water in the reservoir between time  $t_1$  and time  $t_2$ .

2

- 
- If  $C$  is the concentration of the product of a chemical reaction at time  $t$ , then the rate of reaction is the derivative  $C'$ . So

$$\int_{t_1}^{t_2} C'(t) dt = C(t_1) - C(t_2)$$

is the change in the concentration of from time  $t_1$  to time  $t_2$ .

- If the mass of a rod measured from the left end to a point  $x$  is  $m(x)$ , then the linear density is  $\rho(x) = m'(x)$ . So

$$\int_a^b \rho(x) dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between  $x = a$  and  $x = b$ .

- If the rate of growth of a population is  $\frac{dP}{dt}$ , then

$$\int_{t_1}^{t_2} P'(t) dt = P(t_2) - P(t_1)$$

is the net change in population during the time period from  $t_1$  to  $t_2$ . (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

- If an object moves along a straight line with position function  $s(t)$ , then its velocity is  $v(t) = s'(t)$ , so

$$\text{displacement} = \int_{t_1}^{t_2} v(t) dt = s(t_2) - s(t_1)$$

is the net change of position, or displacement, of the particle during the time period from  $t_1$  to  $t_2$ .

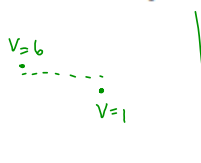
- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when  $v(t) \geq 0$  (the particle moves to the right) and also the intervals when  $v(t) \leq 0$  (the particle moves to the left). In both cases the distance is computed by integrating  $|v(t)|$ , the speed. Therefore

$$\text{distance} = \int_{t_1}^{t_2} |v(t)| dt$$

**Example 2.** A particle moves along a line so that its velocity at time  $t$  is  $v(t) = t^2 - 2t - 8$ .

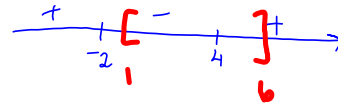
1. Find the displacement of the particle during the time period  $1 \leq t \leq 6$ .

displacement  $\int_1^6 (t^2 - 2t - 8) dt = \left[ \frac{t^3}{3} - t^2 - 8t \right]_1^6 = \frac{6^3}{3} - 6^2 - 8(6) - \frac{1}{3} + 1 + 8 = -\frac{10}{3}$



2. Find the distance traveled during this time period.

where  $v(t) > 0$  ?  $t^2 - 2t - 8 > 0$   
 $(t-4)(t+2) > 0$



$v(t) > 0$  on  $(4, 6)$   
 $v(t) < 0$  on  $(1, 4)$

distance =  $\int_1^6 |v(t)| dt = \int_1^4 |v(t)| dt + \int_4^6 |v(t)| dt$

$$|v(t)| = \begin{cases} v(t), & \text{if } v(t) > 0 \\ -v(t), & \text{if } v(t) < 0 \end{cases}$$

$$|v(t)| = \begin{cases} t^2 - 2t - 8, & 4 < t < 6 \\ -(t^2 - 2t - 8), & 1 < t < 4 \end{cases}$$

$$= - \int_1^4 (t^2 - 2t - 8) dt + \int_4^6 (t^2 - 2t - 8) dt = - \left[ \frac{t^3}{3} - t^2 - 8t \right]_1^4 + \left[ \frac{t^3}{3} - t^2 - 8t \right]_4^6$$

$$= - \left( \frac{4^3}{3} - 4^2 - 8(4) \right) + \left( \frac{1}{3} - 1 - 8 \right) + \left( \frac{6^3}{3} - 6^2 - 8(6) \right) - \left( \frac{4^3}{3} - 4^2 - 8(4) \right)$$

$$= \left[ -\frac{64}{3} + 16 + 32 + \frac{1}{3} - 9 + 36 - 48 - \frac{64}{3} + 16 + 32 \right] = 74 - \frac{127}{3} = \frac{95}{3}$$

**Example 3.** The linear density of a rod of length 4 m is given by  $\rho(x) = 9 + 2\sqrt{x}$  measured in kg/m, where  $x$  is measured in meters from one end of the rod. Find the total mass of the rod.

$$\begin{aligned} \text{total mass} &= \int_0^4 \rho(x) dx = \int_0^4 (9 + 2\sqrt{x}) dx = \left[ 9x + \frac{2x^{1/2+1}}{1/2+1} \right]_0^4 = \left[ 9x + \frac{2}{3} \cdot 2x^{3/2} \right]_0^4 \\ &= \left[ 9x + \frac{4}{3} x^{3/2} \right]_0^4 = 9(4) + \frac{4}{3} \cdot 4^{3/2} = \boxed{36 + \frac{32}{3}} = \frac{140}{3} \end{aligned}$$

**Example 4.** Water flows from the bottom of a storage tank at a rate of  $r(t) = 200 - 4t$  liters per minute, where  $0 \leq t \leq 50$ . Find the amount of water that flows from the tank during the first 10 minutes.

$$v(t) = \int_0^{10} r(t) dt = \int_0^{10} (200 - 4t) dt = \left[ 200t - 2t^2 \right]_0^{10} = 200(10) - 2(100) = 1800 (\text{L})$$