Section 6.5 **The Substitution Rule**

The substitution rule for indefinite integrals If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boxed{\int f(g(x))g'(x)dx = \int f(u)du}$$

$u = g(x)$
$du = g'(x)dx$

Example 1. Evaluate each integral:

1. $\frac{1}{3}\int 3x^2 e^{x^3} dx = \left| \begin{array}{l} u = x^3 \\ du = (x^3)' dx \\ du = 3x^2 dx \end{array} \right| = \frac{1}{3}\int e^{x^3}\underbrace{(3x^2 dx)}_{du} = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C$

$= \boxed{\frac{1}{3}e^{x^3} + C}$

2. $\int \frac{x + \arcsin x}{\sqrt{1 - x^2}} dx = -\frac{1}{2}\int \frac{(-2)x}{\sqrt{1-x^2}} dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx$

$u = 1 - x^2$
$du = -2x\,dx$

$v = \arcsin x$
$dv = \frac{dx}{\sqrt{1-x^2}}$

$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + \int v\,dv$

$= -\frac{1}{2}\,\frac{u^{-1/2+1}}{-1/2+1} + \frac{v^2}{2} + C$

$= -u^{1/2} + \frac{v^2}{2} + C$

$= \boxed{-(1-x^2)^{1/2} + \frac{(\arcsin x)^2}{2} + C}$

3. $\frac{1}{5}\int 5\sin(\underbrace{5x + 4}_{u})dx \quad \left| \begin{array}{l} u = 5x+4 \\ du = (5x+4)' dx \\ du = 5dx \end{array} \right| = \frac{1}{5}\int \sin u\,du = -\frac{1}{5}\cos u + C$

$= \boxed{-\frac{1}{5}\cos(5x+4) + C}$

4. $\frac{2}{3}\int \frac{3\,dx}{2\sqrt{3x + 1}}$ (du) $= \left| \begin{array}{l} v = 3x+1 \\ dv = 3dx \end{array} \right|$

$= \frac{1}{3}\int \frac{dv}{\sqrt{v}} = \frac{1}{3}\,\frac{v^{-1/2+1}}{-1/2+1} + C$

$= \frac{2}{3} v^{1/2} + C$

$= \frac{2}{3}\sqrt{3x+1} + C$

$\left| \begin{array}{l} u = \sqrt{3x+1} \\ du = \frac{3}{2}(3x+1)^{-1/2} dx \\ du = \frac{3}{2\sqrt{3x+1}} dx \end{array} \right| = \frac{2}{3}\int du = \frac{2}{3}u + C$

$= \boxed{\frac{2}{3}\sqrt{3x+1} + C}$

5. $\displaystyle\int \frac{2x^2+4x}{x^3+3x^2-4}dx$

$u = x^3+3x^2-4$
$du = (x^3+3x^2-4)'\,dx$
$du = (3x^2+6x)dx$
$du = 3(x^2+2x)dx$

$$= \int \frac{2\overbrace{(x^2+2x)\,dx}^{\frac{du}{3}}}{\underbrace{x^3+3x^2-4}_{u}}$$

$$= \frac{2}{3}\int \frac{du}{u} = \frac{2}{3}\ln|u| + C = \boxed{\frac{2}{3}\ln|x^3+3x^2-4| + C}$$

6. $\displaystyle\int x^2\sqrt{2+x}\,dx$

$u = 2+x \Rightarrow x = u-2$
$du = dx$

$$= \int \underbrace{(u-2)^2}_{x^2}\underbrace{\sqrt{u}}_{\sqrt{2+x}}\,\overbrace{du}^{dx}$$

$$= \int (u^2-4u+4)\sqrt{u}\,du = \int \left[u^2\sqrt{u} - 4u\sqrt{u} + 4\sqrt{u}\right]du$$

$$= \int \left(u^{5/2} - 4u^{3/2} + 4u^{1/2}\right)du = \frac{u^{5/2+1}}{5/2+1} - 4\frac{u^{3/2+1}}{3/2+1} + 4\frac{u^{1/2+1}}{1/2+1} + C$$

$$= \frac{2}{7}u^{7/2} - \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C = \boxed{\frac{2}{7}(x+2)^{7/2} - \frac{8}{5}(x+2)^{5/2} + \frac{8}{3}(x+2)^{3/2} + C}$$

7. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

$$\int x^2 \cdot \overbrace{x\sqrt{x^2+1}\,dx}^{\frac{du}{2}}$$

$u = x^2+1 \Rightarrow x^2 = u-1$
$du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$$= \int \underbrace{(u-1)}_{x^2}\underbrace{\sqrt{u}}_{\sqrt{x^2+1}}\underbrace{\frac{du}{2}}_{x\,dx}$$

$$= \frac{1}{2}\int (u\sqrt{u} - \sqrt{u})\,du = \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{2}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + C = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{5}(x^2+1)^{5/2} - \frac{\cancel{2}}{3}(x^2+1)^{3/2}\right) + C$$

The substitution rule for definite integrals If $g'(x)$ is continuous on $[a,b]$ and f is continuous on the range of g, then

$$\boxed{\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du}$$

Example 2. Evaluate the integral:

1. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \left| \begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \\ e \to \ln e = 1 \\ e^4 \to \ln e^4 = 4 \end{array} \right| = \int_1^4 \frac{1}{\sqrt{u}}du = \int_1^4 u^{-1/2}du = \frac{u^{-1/2+1}}{-1/2+1}\Big]_1^4$

$= 2u^{1/2}\Big]_1^4 = 2(\sqrt{4} - \sqrt{1}) = \boxed{2}$

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$\boxed{\int \sqrt{x^2+a} dx = \ln|x+\sqrt{x^2+a}| + C}$

$x^4 = (x^2)^2$

2. $\displaystyle\frac{1}{2}\int_0^1 \frac{2x dx}{\sqrt{1+x^4}} = \left| \begin{array}{l} u = x^2 \\ du = 2x dx \\ 0 \to 0^2 = 0 \\ 1 \to 1^2 = 1 \end{array} \right| = \frac{1}{2}\int_0^1 \frac{du}{\sqrt{1+u^2}} = \frac{1}{2}\ln\left|u + \sqrt{u^2+1}\right|\Big]_0^1$

$= \frac{1}{2}\left[\ln|1+\sqrt{1+1}| - \ln|0+\sqrt{0+1}|\right] = \boxed{\frac{1}{2}\ln(1+\sqrt{2})}$

3. $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}dz \quad \left| \begin{array}{l} u = e^z + z \\ du = (e^z+1)dz \\ z=0 \Rightarrow u = e^0+0 = 1 \\ z=1 \Rightarrow u = e^1+1 = e+1 \end{array} \right| = \int_1^{e+1} \frac{du}{u} = \ln|u|\Big]_1^{e+1}$

$= \ln(e+1) - \ln 1 = \boxed{\ln(e+1)}$

Integrals of symmetric functions Suppose f is continuous on $[-a, a]$.

(a) If f is **even**, that is $f(-x) = f(x)$, then $\boxed{\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx}$

(b) If f is **odd**, that is $f(-x) = -f(x)$, then $\boxed{\int_{-a}^{a} f(x)dx = 0}$

$y = f(x)$, f is even

area (A_1) = area (A_2)

area $(A_1) = \int_{-a}^{0} f(x)dx$

area $(A_2) = \int_{0}^{a} f(x)dx$

$\int_{-a}^{a} f(x)dx =$ area (A_1) + area (A_2)

$= 2 \cdot$ area$(A_2) = 2\int_0^a f(x)dx$

f is odd

$\int_{-a}^{a} f(x)dx =$ area (A_2) $-$ area (A_1) $= 0$

area (A_1) = area (A_2)

Example 4. Evaluate the integral $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6} dx$.

even functions: $y = x^2, x^4, x^6, \ldots$
$y = \cos x$

odd functions: $y = x, x^3, x^5, \ldots$
$y = \sin x$
$y = \arctan x$

is $\frac{x^2 \sin x}{1+x^6}$ even or odd?

$\frac{x^2}{1+x^6}$ even

$\sin x$ odd

(even)(odd) = odd

$\frac{x^2 \sin x}{1+x^6}$ is odd

$\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} dx = \boxed{0}$