Slopes and tangents to parametric curves

Suppose that the curve C is given by parametric equations $x = x(t)$, $y = y(t)$, then

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}}$$

Example 1. Find an equation of the tangent line to the curve $x(t) = t\sin t$, $y(t) = t\cos t$ at the point corresponding to $t = \pi$.

Tangent line: $y - \underbrace{y(\pi)}_{-\pi} = \frac{dy}{dx}(\pi)\,[x - \underbrace{x(\pi)}_{0}]$

$x(\pi) = \pi \underbrace{\sin \pi}_{0} = 0$

$y(\pi) = \pi \underbrace{\cos \pi}_{-1} = -\pi$

slope of the tangent line $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(t\cos t)'}{(t\sin t)'}$

$$= \frac{t'\cos t + t(\cos t)'}{t'\sin t + t(\sin t)'} = \frac{\cos t - t\sin t}{\sin t + t\cos t}$$

slope when $t = \pi$ is $\frac{dy}{dx}(\pi) = \frac{\cos\pi - \pi\sin\pi}{\sin\pi + \pi\cos\pi} = \frac{-1}{-\pi} = \frac{1}{\pi}$

tangent line: $y - (-\pi) = \frac{1}{\pi}(x - 0)$

$$\boxed{y + \pi = \frac{x}{\pi}}$$

$x = t^4 - 3t, \quad y = 3t^3 - 9$

Example 2. Find the points on the curve $x = t(t^3 - 3)$, $y = 3(t^3 - 3)$, where the tangent is vertical or horizontal.

slope $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{(3t^3-9)'}{(t^4-3t)'} = \dfrac{9t^2}{4t^3-3}$

horizontal tangent $y'(t)=0$ $\Rightarrow$ slope $=0$ $\quad \dfrac{9t^2}{4t^3-3} = 0$ or $9t^2 = 0$

$t = 0$

point on the curve $(x(0), y(0)) = \boxed{(0, -9)\text{ - horizontal tangent}}$

vertical tangent $x'(t)=0$ $\Rightarrow$ $4t^3 - 3 = 0$ or $t^3 = \frac{3}{4}$, $t = \sqrt[3]{\frac{3}{4}}$

point on the curve: $x\left(\sqrt[3]{\frac{3}{4}}\right) = \sqrt[3]{\frac{3}{4}}\left(\left(\sqrt[3]{\frac{3}{4}}\right)^3 - 3\right) = \sqrt[3]{\frac{3}{4}}\left(-\frac{9}{4}\right)$

$y\left(\sqrt[3]{\frac{3}{4}}\right) = 3\left(\frac{3}{4} - 3\right) = -\frac{27}{4}$

$\boxed{\left(-\frac{9}{4}\left(\sqrt[3]{\frac{3}{4}}\right), -\frac{27}{4}\right)\text{ vertical tangent}}$

Example 3. At what points on the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent parallel to the line with the equations $x = -7t$, $y = 12t - 5$?

line : $x = -7t$, $y = 12t - 5$, slope $= -\frac{12}{7}$

$t = -\frac{x}{7} \Rightarrow y = -\frac{12x}{7} - 5$

$x = t^3 + 4t$

$y = 6t^2$

slope of a tangent line is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(6t^2)'}{(t^3+4t)'} = \frac{12t}{3t^2+4} = -\frac{12}{7}$

$\frac{t}{3t^2+4} = -\frac{1}{7}$

$7t = -3t^2 - 4$

$3t^2 + 7t + 4 = 0$

$t_1 = \frac{-7 + \sqrt{7^2 - 4(4)(3)}}{6} = \frac{-7+1}{6} = -1$

$t_2 = \frac{-7-1}{6} = -\frac{8}{6} = -\frac{4}{3}$

points on the curve

$(x(-1), y(-1)) = (-5, 6)$

$\left(x\left(-\frac{4}{3}\right), y\left(-\frac{4}{3}\right)\right) = \left(-\frac{208}{27}, \frac{32}{3}\right)$

$\left(-\frac{4}{3}\right)^3 + 4\left(-\frac{4}{3}\right) = \frac{-64}{27} - \frac{16}{3} = \frac{-208}{27}$

$6\left(\frac{16}{9}\right) = \frac{2(16)}{3}$