Vector functions.

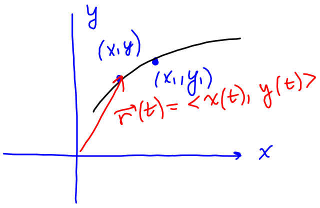
Definition. The curve of a type $\boxed{x = x(t),\ y = y(t)}$ is called a **parametric curve** and the variable t is called a **parameter**.

Definition. Vector

$$\boxed{\mathbf{r}(t) = < x(t), y(t) > = x(t)\mathbf{i} + y(t)\mathbf{j}}$$

is called the **position vector** for the point with coordinates $(x(t), y(t))$.

A function such as $\mathbf{r}(t)$, whose range is a set of vectors, is called a **vector function** of t.



$x = x(t)$ $\quad a \le t \le b.$

$y = y(t)$

$t = t_1 \rightarrow \quad x_1 = x(t_1)$

$y_1 = y(t_1)$

Example 1.

1. Sketch the curve represented by the parametric equation $x(t) = \cancel{\frac{1-t}{t}}$, $\cancel{y = t^2}$.

$x = 2\sin\theta$
$y = 3\cos\theta$, $\quad 0 \le \theta \le \pi$

Eliminate the parameter θ:

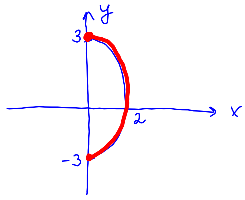
$x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$

$y = 3\cos\theta \Rightarrow \cos\theta = \frac{y}{3}$

$\boxed{\cos^2 x + \sin^2 x = 1}$

$\underbrace{\sin^2\theta + \cos^2\theta}_{1} = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2$

$1 = \frac{x^2}{4} + \frac{y^2}{9}$ ellipse



$\begin{cases} x(t) = \cos t \\ y(t) = \cos 2t \end{cases}$, $\quad 0 \le t \le 2\pi$

$-1 \le |x| \le 1$
$-1 \le |y| \le 1$

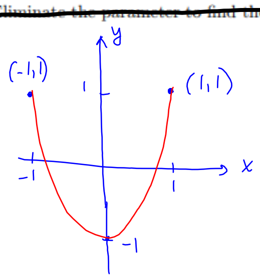
$t = 0$: $x(0) = \cos 0 = 1$, $y(0) = \cos 0 = 1$

$t = 2\pi$: $x(2\pi) = \cos 2\pi = 1$, $y(2\pi) = \cos 4\pi = 1$

$t = \pi$: $x(\pi) = \cos\pi = -1$, $y(\pi) = \cos 2\pi = 1$

$t = \frac{\pi}{2}$: $x(\frac{\pi}{2}) = 0$, $y(\frac{\pi}{2}) = \cos\pi = -1$

~~2. Eliminate the parameter to find the Cartesian equation of the curve.~~



eliminate t:

$y(t) = \boxed{\cos 2t = 2\overbrace{\cos^2 t}^{x^2} - 1}$

$y = 2x^2 - 1$

Example 2. An object is moving in the xy-plane and its position after t seconds is $\mathbf{r}(t) =< t-3, t^2-2t >$.

1. Find the position of the object at time $t = 5$.

$\vec{r}(5) = <5-3,\ 5^2-2(5)> = \boxed{<2, 15>}$

2. At what time is the object at the point (1,8).

$<1,8> = <t-3,\ t^2-2t>$

or $\begin{cases} t-3=1 \Rightarrow t=4 \\ t^2-2t=8 \end{cases}$

plug $t=4$ into the 2nd equation

$4^2-2(4) = 16-8=8$

$\boxed{t=4}$

3. Does the object pass through the point (3,20).

$<3,20> = <t-3,\ t^2-2t>$

$\begin{cases} t-3=3 \Rightarrow t=6 \\ t^2-2t=20 \end{cases}$

Plug $t=6$ into the 2nd equation

$6^2-2(6) = 36-12 = 24 \neq 20$

$\boxed{NO}$

4. Find an equation in x and y whose graph is the path of the object. Eliminate t.

$x = t-3 \Rightarrow t=x+3$

$y = t^2-2t$

$\boxed{y=(x+3)^2-2(x+3)}$

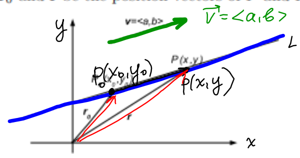
$x^2+6x+9-2x-6$

$\boxed{y=x^2+4x+3}$

Vector equation of a line.

A line L is determined by a point P_0 on L and a direction. Let $\mathbf{v}$ be a vector parallel to line L. Let P be be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P and P_0.

$\vec{v} = <a,b>$ parallel to the line



$\overrightarrow{P_0P}$ and $\vec{v}$ are parallel

there is a constant t such that

$\overrightarrow{P_0P} = t\vec{v}$

Then the vector equation of line L is

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}}$$

If $\mathbf{r} =< x(t), y(t) >$, $\mathbf{v} =< a, b >$ and $P(x_0, y_0)$ then parametric equations of the line L are

$$\boxed{x(t) = x_0 + at,} \quad \boxed{y(t) = y_0 + bt}$$

Example 3. Find a vector, parametric, and Cartesian equations for the line containing the point (2,-1) and parallel to $2\mathbf{i} + 3\mathbf{j}$. $= <2,3>$

vector equation $\boxed{\vec{r} = <2,-1> + t<2,3>}$

$<x,y> = <2,-1> + t<2,3>$

$<x,y> = <2+2t, -1+3t>$

parametric equations: $\boxed{\begin{aligned} x &= 2+2t \\ y &= -1+3t \end{aligned}}$

Eliminate t: $x = 2+2t \Rightarrow t = \frac{x-2}{2}$

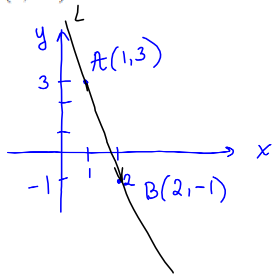
$2y = \left(-1 + 3 \cdot \frac{x-2}{2}\right)2$

$2y = -2 + 3(x-2)$

$\boxed{3x-2y-8=0}$ cartesian equation

$\vec{n} = <3,-2>$ is perpendicular to the lin

Example 4. Find a vector and parametric equations for the line passing through the points $A(1, 3)$ and $B(2, -1)$.



$\overrightarrow{AB} = \langle 2-1, -1-3 \rangle = \langle 1, -4 \rangle$

vector equation: $\vec{r} = \langle 1, 3 \rangle + t\langle 1, -4 \rangle$ or $\vec{r} = \langle 2, -1 \rangle + t\langle 1, -4 \rangle$

parametric equations: $\begin{cases} x = 1 + t \\ y = 3 - 4t \end{cases}$ | $\begin{cases} x = 2 + t \\ y = -1 - 4t \end{cases}$