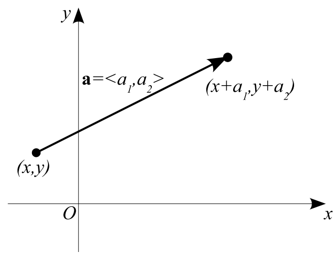
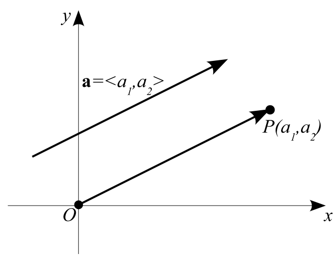
Chapter 1. **Introduction to vectors and vector functions**
Section 1.1 **Vectors**

Definition. A **two-dimensional vector** is an ordered pair $\mathbf{a} =< a_1, a_2 >$ of real numbers. The numbers a_1 and a_2 are called the **components** of **a**.

A **representation** of the vector $\mathbf{a} =< a_1, a_2 >$ is a directed line segment $\overrightarrow{AB}$ from any point $A(x, y)$ to the point $B(x + a_1, y + a_2)$.



A particular representation of $\mathbf{a} =< a_1, a_2 >$ is the directed line segment $\overrightarrow{OP}$ from the origin to the point $P(a_1, a_2)$, and $\mathbf{a} =< a_1, a_2 >$ is called the **position vector** of the point $P(a_1, a_2)$.



Given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$\overrightarrow{AB} =< x_2 - x_1, y_2 - y_1 >$$

Example 1. Find a vector **a** with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

(a) $A(1, 2)$, $B(3, 3)$;

(b) $A(1, -2)$, $B(-2, 3)$.

The **magnitude (length)** $|\mathbf{a}|$ of $\mathbf{a}$ is the length of any its representation.
The length of $\mathbf{a} =< a_1, a_2 >$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

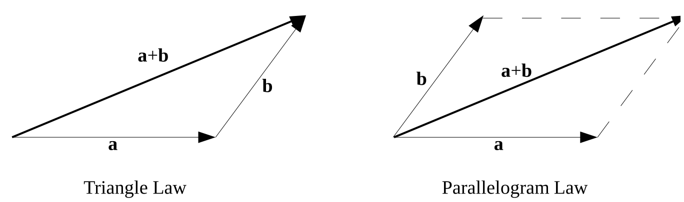
The length of the vector $\overrightarrow{AB}$ from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The only vector with length 0 is the **zero vector** $\mathbf{0} =< 0, 0 >$. This vector is the only vector with no specific direction.

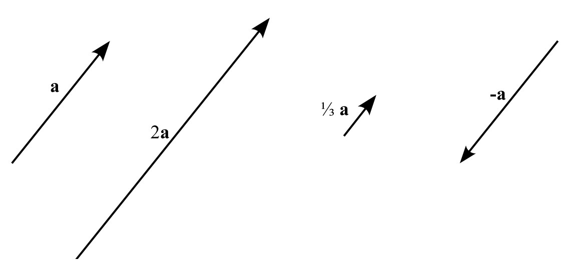
Example 2. Find the length of the vectors from Example 1.

Vector addition If $\mathbf{a} =< a_1, a_2 >$ and $\mathbf{b} =< b_1, b_2 >$, then the vector $\mathbf{a} + \mathbf{b}$ is defined by $\mathbf{a} + \mathbf{b} =< a_1 + b_1, a_2 + b_2 >$.



Multiplication of a vector by a scalar If c is a scalar and $\mathbf{a} =< a_1, a_2 >$, then the vector is defined by

$$c\mathbf{a} =< ca_1, ca_2 >$$



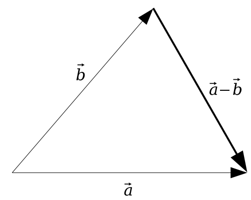
$$|c\mathbf{a}| = c|\mathbf{a}|$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are called **parallel** if $\mathbf{b} = c\mathbf{a}$ for some scalar c.

By the **difference** of two vectors, we mean

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

so, if $\mathbf{a} =< a_1, a_2 >$ and $\mathbf{b} =< b_1, b_2 >$, then $\mathbf{a} - \mathbf{b} =< a_1 - b_1, a_2 - b_2 >$.



Example 3. If $\mathbf{a} =< -1, 2 >$ and $\mathbf{b} =< -2, -1 >$, find

(a) $\mathbf{a} + \mathbf{b}$

(b) $1/2\mathbf{b}$

(c) $\mathbf{a} - \mathbf{b}$

(d) $|2\mathbf{a} - 5\mathbf{b}|$

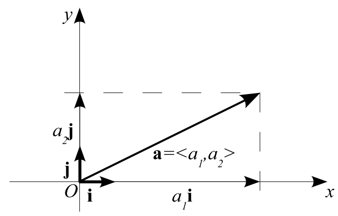
Properties of vectors. If **a**, **b**, and **c** are vectors and k and m are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \vec{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
6. $(k + m)\mathbf{a} = k\mathbf{a} + m\mathbf{a}$
7. $(km)\mathbf{a} = k(m\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Let $\mathbf{i} =< 1, 0 >$ and $\mathbf{i} =< 0, 1 >$.

$$|\mathbf{i}| = |\mathbf{i}| = 1$$

$\mathbf{a} =< a_1, a_2 >= a_1\mathbf{i} + a_2\mathbf{i}$



Example 4. Express $\mathbf{a} =< 2, 4 >$, $\mathbf{b} =< -1, 3 >$, and $2\mathbf{a} + \mathbf{b}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

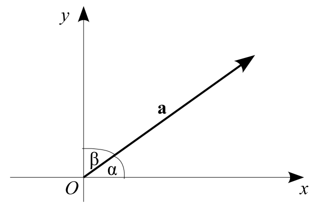
A **unit vector** is a vector whose length is 1.
A vector

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \left\langle \frac{a_1}{|\mathbf{a}|}, \frac{a_2}{|\mathbf{a}|} \right\rangle$$

is a unit vector that has the same direction as $\mathbf{a} =< a_1, a_2 >$.

Example 5. Given vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{i}$, $\mathbf{b} =< -2, 3 >$. Find a unit vector $\mathbf{u}$ that has the same direction as $2\mathbf{b} + \mathbf{a}$.

Direction angles and direction cosines. The **direction angles** of a nonzero vector **a** are the angles α and β in the interval $[0, \pi]$ that **a** makes with the positive $x-$ and $y-$ axes. The cosines of these direction angles, $\cos\alpha$ and $\cos\beta$ are called the **direction cosines** of the vector **a**.



$$\cos\alpha = \frac{a_1}{|\mathbf{a}|}, \quad \cos\beta = \frac{a_2}{|\mathbf{a}|}, \qquad \cos^2\alpha + \cos^2\beta = 1$$

We can write

$$\mathbf{a} =< a_1, a_2 >= |\mathbf{a}| < \cos\alpha, \cos\beta >$$

Therefore

$$\frac{1}{|\mathbf{a}|}\mathbf{a} =< \cos\alpha, \cos\beta >$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

Example 6. Let **c** be the vector obtained by rotating $\mathbf{a} =< 1, 3 >$ by an angle of 60 degrees in the counterclockwise direction. Compute the vector **c**.

Example 7. Two forces $\vec{F_1}$ and $\vec{F_2}$ with magnitudes 10 lb and 12 lb act on an object at a point P as shown in the figure. Find the resultant force $\vec{F}$ acting at P as well as its magnitude and its direction.