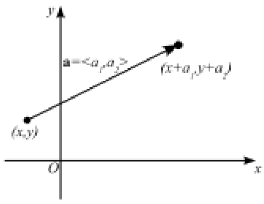
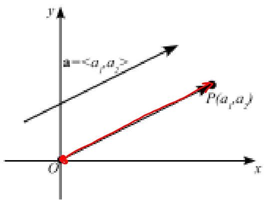
[redacted] Introduction to vectors and vector functions
[redacted] Vectors

Definition. A **two-dimensional vector** is an ordered pair $\mathbf{a} = < a_1, a_2 >$ of real numbers. The numbers a_1 and a_2 are called the **components** of $\mathbf{a}$.

A **representation** of the vector $\mathbf{a} = < a_1, a_2 >$ is a directed line segment $\overrightarrow{AB}$ from any point $A(x, y)$ to the point $B(x + a_1, y + a_2)$.



A particular representation of $\mathbf{a} = < a_1, a_2 >$ is the directed line segment $\overrightarrow{OP}$ from the origin to the point $P(a_1, a_2)$, and $\mathbf{a} = < a_1, a_2 >$ is called the **position vector** of the point $P(a_1, a_2)$.



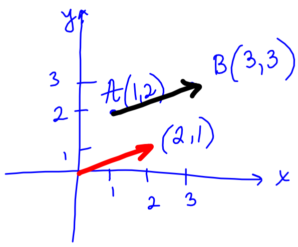
Given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$\overrightarrow{AB} = < x_2 - x_1, y_2 - y_1 >$$

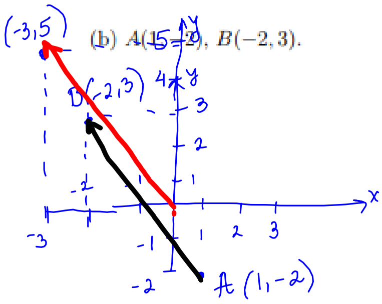
Example 1. Find a vector $\mathbf{a}$ with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

(a) $A(1, 2)$, $B(3, 3)$;

components of $\overrightarrow{AB} = < 3-1, 3-2 > = < 2, 1 >$



(b) $A(1, -2)$, $B(-2, 3)$.



components of $\overrightarrow{AB} = < -2-1, 3-(-2) >$
$= < -3, 5 >$

The **magnitude (length)** $|\mathbf{a}|$ of $\mathbf{a}$ is the length of any its representation.
The length of $\mathbf{a} =< a_1, a_2 >$ is

$$\boxed{|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}}$$

The length of the vector $\overrightarrow{AB}$ from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\boxed{|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

The only vector with length 0 is the **zero vector** $\mathbf{0} =< 0, 0 >$. This vector is the only vector with no specific direction.

Example 2. Find the length of the vectors from Example 1.

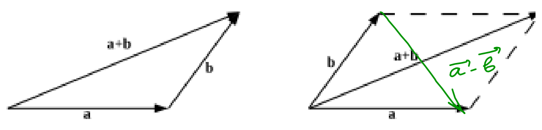
(a) $\overrightarrow{AB} = <2,1>$

$|\overrightarrow{AB}| = \sqrt{2^2+1^2} = \sqrt{5}$

(b) $\overrightarrow{AB} = <-3,5>$

$|\overrightarrow{AB}| = \sqrt{(-3)^2 + 5^2} = \sqrt{25+9} = \sqrt{34}$

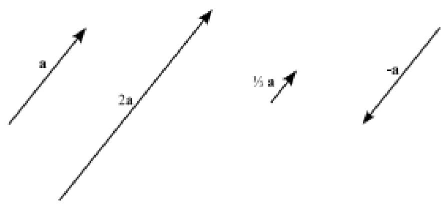
Vector addition If $\mathbf{a} =< a_1, a_2 >$ and $\mathbf{b} =< b_1, b_2 >$, then the vector $\mathbf{a} + \mathbf{b}$ is defined by $\mathbf{a} + \mathbf{b} =< a_1 + b_1, a_2 + b_2 >$.



Triangle Law Parallelogram Law

Multiplication of a vector by a scalar If c is a scalar and $\mathbf{a} =< a_1, a_2 >$, then the vector is defined by

$$c\mathbf{a} =< ca_1, ca_2 >$$



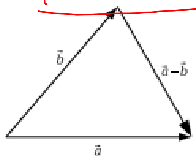
$$\boxed{|c\mathbf{a}| = c|\mathbf{a}|}$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are called **parallel** if $\mathbf{b} = c\mathbf{a}$ for some scalar c.

By the **difference** of two vectors, we mean

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

so, if $\mathbf{a} =< a_1, a_2 >$ and $\mathbf{b} =< b_1, b_2 >$, then $\boxed{\mathbf{a} - \mathbf{b} =< a_1 - b_1, a_2 - b_2 >}$.



Example 3. If $\mathbf{a} =< -1, 2 >$ and $\mathbf{b} =< -2, -1 >$, find

(a) $\mathbf{a} + \mathbf{b}$ $= <-1,2> + <-2,-1> = <-1-2, 2-1> = \boxed{<-3,1>}$

(b) $1/2\mathbf{b}$ $= \frac{1}{2} <-2,-1> = \boxed{<-1, -\frac{1}{2}>}$

(c) $\mathbf{a} - \mathbf{b}$ $= <-1,2> - <-2,-1> = <-1-(-2), 2-(-1)> = \boxed{<1,3>}$

(d) $|2\mathbf{a} - 5\mathbf{b}|$

$2\vec{a} - 5\vec{b} = 2<-1,2> - 5<-2,-1> = <2(-1)-5(-2), 2(2)-5(-1)>$

$= <8, 9>$

$|<8,9>| = \sqrt{8^2+9^2} = \sqrt{64+81} = \boxed{\sqrt{145}}$

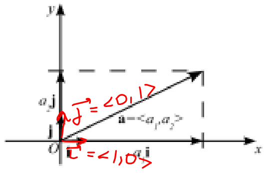
Properties of vectors. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and k and m are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \vec{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
6. $(k + m)\mathbf{a} = k\mathbf{a} + m\mathbf{a}$
7. $(km)\mathbf{a} = k(m\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Let $\mathbf{i} =< 1,0 >$ and $\vec{j} =< 0,1 >$.

$$|\mathbf{i}| = |\vec{j}| = 1$$

$\mathbf{a} =< a_1, a_2 >= a_1\mathbf{i} + a_2\vec{j}$



Example 4. Express $\mathbf{a} =< 2,4 >$, $\mathbf{b} =< -1,3 >$, and $2\mathbf{a} + \mathbf{b}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

$\vec{a} = <2,4> = \boxed{2\vec{i} + 4\vec{j}}$

$\vec{b} = <-1,3> = \boxed{-\vec{i} + 3\vec{j}}$

$2\vec{a} + \vec{b} = 2<2,4> + <-1,3> = <4-1, 8+3> = <3,11> = \boxed{3\vec{i} + 11\vec{j}}$

A **unit vector** is a vector whose length is 1. $\vec{i}$ and $\vec{j}$ are unit.

A vector

$\vec{u}$, $|\vec{u}|=1$, $\vec{a}$

$$\boxed{\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \left\langle \frac{a_1}{|\mathbf{a}|}, \frac{a_2}{|\mathbf{a}|} \right\rangle}$$

is a unit vector that has the same direction as $\mathbf{a} =< a_1, a_2 >$.

Example 5. Given vectors $\mathbf{a} = \mathbf{i} - 2\vec{j}$, $\mathbf{b} =< -2,3 >$. Find a unit vector $\mathbf{u}$ that has the same direction as $2\mathbf{b} + \mathbf{a}$.

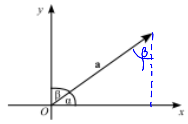
$\vec{a} = <1,-2>$, $\vec{b} = <-2,3>$

$\vec{c} = 2\vec{b} + \vec{a} = <1,-2> + 2<-2,3> = <1-4, -2+6> = <-3,4>$

$|<-3,4>| = \sqrt{9+16} = \sqrt{25} = 5$

$\vec{u} = \dfrac{<-3,4>}{|<-3,4>|} = \dfrac{<-3,4>}{5} = \boxed{<-\frac{3}{5}, \frac{4}{5}>}$

Direction angles and direction cosines. The **direction angles** of a nonzero vector **a** are the angles α and β in the interval $[0, \pi]$ that **a** makes with the positive $x-$ and $y-$ axes. The cosines of these direction angles, $\cos\alpha$ and $\cos\beta$ are called the **direction cosines** of the vector **a**.



$\beta = 90 - \alpha$

$\cos\beta = \cos(90-\alpha) = \sin\alpha$

$$\cos\alpha = \frac{a_1}{|\mathbf{a}|}, \quad \cos\beta = \frac{a_2}{|\mathbf{a}|}, \qquad \cos^2\alpha + \cos^2\beta = 1$$

We can write

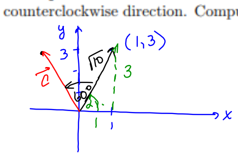
$$\boxed{\mathbf{a} =< a_1, a_2 >= |\mathbf{a}| < \cos\alpha, \cos\beta > \; = \; |\vec{a}| < \cos\alpha, \sin\alpha >}$$

Therefore

$$\frac{1}{|\mathbf{a}|}\mathbf{a} =< \cos\alpha, \cos\beta >$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

Example 6. Let **c** be the vector obtained by rotating **a** $=< 1, 3 >$ by an angle of 60 degrees in the counterclockwise direction. Compute the vector **c**.



$|\vec{a}| = |\vec{c}| = \sqrt{1+9} = \sqrt{10}$

$\cos\alpha = \frac{1}{\sqrt{1+9}} = \frac{1}{\sqrt{10}}, \quad \sin\alpha = \frac{3}{\sqrt{10}}$

$\vec{c} = |\vec{c}| < \cos(\alpha + 60°), \sin(\alpha + 60°) >$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

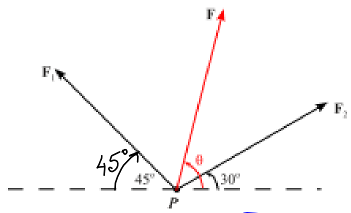
$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(\alpha + 60°) = \cos\alpha \cos 60° - \sin\alpha \sin 60° = \frac{1}{\sqrt{10}} \cdot \frac{1}{2} - \frac{3}{\sqrt{10}} \frac{\sqrt{3}}{2} = \frac{1}{2\sqrt{10}} - \frac{3\sqrt{3}}{2\sqrt{10}} = \frac{1-3\sqrt{3}}{2\sqrt{10}}$

$\sin(\alpha + 60°) = \sin\alpha \cos 60° + \cos\alpha \sin 60° = \frac{3}{\sqrt{10}} \frac{1}{2} + \frac{1}{\sqrt{10}} \frac{\sqrt{3}}{2} = \frac{3}{2\sqrt{10}} + \frac{\sqrt{3}}{2\sqrt{10}} = \frac{3+\sqrt{3}}{2\sqrt{10}}$

$\vec{c} = \cancel{\sqrt{10}} < \frac{1-3\sqrt{3}}{2\cancel{\sqrt{10}}}, \frac{3+\sqrt{3}}{2\cancel{\sqrt{10}}} > = \boxed{< \frac{1-3\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2} >}$

Example 7. Two forces $\vec{F_1}$ and $\vec{F_2}$ with magnitudes 10 lb and 12 lb act on an object at a point P as shown in the figure. Find the resultant force $\vec{F}$ acting at P as well as its magnitude and its direction.



$\vec{F} = \vec{F_1} + \vec{F_2}$

$|\vec{F_1}| = 10$

$|\vec{F_2}| = 12$

$\vec{F_2} = |\vec{F_2}| < \cos 30°, \sin 30° > = 12 < \frac{\sqrt{3}}{2}, \frac{1}{2} > = < 6\sqrt{3}, 6 >$

$\vec{F_1} = |\vec{F_1}| < -\cos 45°, \sin 45° > = 10 < -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} > = < -5\sqrt{2}, 5\sqrt{2} >$

resultant force $\vec{F} = < 6\sqrt{3}, 6 > + < -5\sqrt{2}, 5\sqrt{2} > = \boxed{< 6\sqrt{3} - 5\sqrt{2}, 6 + 5\sqrt{2} >}$

$|\vec{F}| = \sqrt{(6\sqrt{3} - 5\sqrt{2})^2 + (6 + 5\sqrt{2})^2}$

$\boxed{\begin{array}{l}(a+b)^2 = a^2 + 2ab + b^2 \\ (a-b)^2 = a^2 - 2ab + b^2\end{array}}$

$= \sqrt{(6\sqrt{3})^2 - 2(6\sqrt{3})(5\sqrt{2}) + (5\sqrt{2})^2 + 6^2 + 2(6)(5\sqrt{2}) + (5\sqrt{2})^2}$

$= \sqrt{36(3) - 2(30)\sqrt{6} + 25(2) + 36 + 2(30)(\sqrt{2}) + 25(2)}$

$\boxed{|\vec{F}| = \sqrt{244 - 60\sqrt{6} + 60\sqrt{2}}}$

$\boxed{\cos\theta = \frac{6\sqrt{3} - 5\sqrt{2}}{|\vec{F}|}}$