Math151 528-530 Spring 2007
REVIEW BEFORE FINAL

VECTORS. THE DOT PRODUCT

Let c be a scalar and $\vec{a} =< a_1, a_2 >= a_1\vec{\imath}+a_2\vec{\jmath}$ and $\vec{b} =< b_1, b_2 >= b_1\vec{\imath}+b_2\vec{\jmath}$ be vectors. Then

$c\vec{a} =< ca_1, ca_2 >$

$\vec{a}+\vec{b} =< a_1 + b_1, a_2 + b_2 >$,

$\vec{a}-\vec{b} =< a_1 - b_1, a_2 - b_2 >$,

the *magnitude* of $\vec{a}$ is $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$,

vector $\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|} \right\rangle$ is a unit vector that has the same direction as $\vec{a}$,

the *dot product* of vectors $\vec{a}$ and $\vec{b}$ is a number $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = a_1b_1 + a_2b_2$,

the *scalar projection* of $\vec{b}$ onto $\vec{a}$ is a number $\text{comp}_{\vec{a}}\vec{b} = \left|\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right|$,

the *vector projection* of $\vec{b}$ onto $\vec{a}$ is a vector $\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|}$.

Given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then $\vec{AB} =< x_2 - x_1, y_2 - y_1 >$.

Example 1 Given vectors $\vec{a} = \vec{\imath} - 2\vec{\jmath}$, $\vec{b} =< -2, 3 >$. Find

(a) a unit vector $\vec{u}$ that has the same direction as $2\vec{b}+\vec{a}$.

$2\vec{b}+\vec{a} = 2 < -2, 3 > + < 1, -2 >=< -4 + 1, 6 - 2 >=< -3, 4 >$,

$|2\vec{b}+\vec{a}| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = 5$.

Then

$\vec{u} = \frac{2\vec{b}+\vec{a}}{|2\vec{b}+\vec{a}|} = \frac{<-3,4>}{5} = \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle$.

(b) $\text{comp}_{\vec{b}}\vec{a}$, $\text{proj}_{\vec{b}}\vec{a}$.

$\vec{a}\cdot\vec{b} = (1)(-2) + (-2)(3) = -2 - 6 = -8$,

$|\vec{b}| = \sqrt{(-2)^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$.

Then

$\text{comp}_{\vec{b}}\vec{a} = \left|\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right| = \left|\frac{-8}{\sqrt{13}}\right| = \frac{8}{\sqrt{13}}$,

$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\frac{\vec{b}}{|\vec{b}|} = \frac{-8}{\sqrt{13}}\frac{<-2,3>}{\sqrt{13}} = \left\langle \frac{16}{13}, \frac{-24}{13} \right\rangle$.

VECTOR EQUATION OF A LINE

A line L is determined by a point P_0 on L and a direction. Let $\vec{v}$ be a vector parallel to line L. Let P be be an arbitrary point on L and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of P and P_0. Then the *vector equation* of line L is

$\vec{r}(t) = \vec{r_0} + t\vec{v}$.

If $\vec{r} =< x(t), y(t) >$, $\vec{v} =< a, b >$ and $P(x_0, y_0)$ then *parametric equations* of the line L are

$x(t) = x_0 + at,\ y(t) = y_0 + bt.$

Example 2 Find a vector and parametric equations for the line passing through the points $A(1, -3)$ and $B(2, 1)$.

Vector $\vec{AB} =< 2-1, 1-(-3) >=< 1, 4 >$ is parallel to line. We can use any point lying on the line, for example $B(2, 1)$, for the point P_0. Then the vector equation of the line becomes

$\vec{r}(t) = \vec{r_0} + t\vec{AB} =< 2, 1 > +t < 1, 4 >=< 2 + t, 1 + 4t >.$

Parametric equations of this line are

$x(t) = 2 + t,\ y(t) = 1 + 4t.$

CONTINUITY. VERTICAL AND HORIZONTAL ASYMPTOTES

Function f is *continuous at a number* a if $\lim\limits_{x\to a} f(x) = f(a)$.

If f is not continuous at a, then f has *discontinuity* at a.

If $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$, then f has a *jump discontinuity* at a,

if either $\lim\limits_{x\to a^+} f(x) = \infty$ or $\lim\limits_{x\to a^-} f(x) = \infty$, then f has an *infinity discontinuity* at a and we say line $x = a$ is a *vertical asymptote* of the curve $y = f(x)$.

and if $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) \neq f(a)$, then f has a *removable discontinuity* at a.

If $\lim\limits_{x\to\infty} f(x) = b$, then we say line $y = b$ is *horizontal asymptote* of the curve $y = f(x)$.

Example 3

(a) Find and classify all points of discontinuity for the function

$$f(x) = \begin{cases} x^2 + 1 & , \quad \text{if } x < 2, \\ x + 2 & , \quad \text{if } x \geq 2. \end{cases}$$

Function f is continuous for all $x \neq 2$. We have to check if f is continuous at $x = 2$.

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2}(x^2 + 1) = 5,$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2}(x + 2) = 4.$

Since $\lim\limits_{x\to 2^+} f(x) \neq \lim\limits_{x\to 2^-} f(x)$, then f has a jump discontinuity at $x = 2$.

(b) Find the vertical and horizontal asymptotes of the curve $y = \frac{x^2+4}{3x^2-3}$.

Function f has infinity discontinuity at $x = 1$ and $x = -1$, this means that curve $y = \frac{x^2+4}{3x^2-3}$ has two vertical asymptotes $x = 1$ and $x = -1$.

Let's now find

$\lim_{x\to\infty} \frac{x^2+4}{3x^2-3} = \lim_{x\to\infty} \frac{x^2(1+\frac{4}{x^2})}{x^2(3-\frac{3}{x^2})} = \lim_{x\to\infty} \frac{1+\frac{4}{x^2}}{3-\frac{3}{x^2}} = \frac{1}{3}.$

Line $y = \frac{1}{3}$ is a horizontal asymptote for the given curve.

DERIVATIVES

Table of derivatives

$1.(C)' = 0,\ C$ is a constant,	$11.(\tan x)' = \sec^2 x,$
$2.(x)' = 1,$	$12.(\cot x)' = -\csc^2 x,$
$3.(x^2)' = 2x,$	$13.(\sec x)' = \sec x \tan x,$
$4.(x^n)' = nx^{n-1},$	$14.(\csc x)' = -\csc x \cot x,$
$5.(\ln x)' = \frac{1}{x},$	$15.(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}},$
$6.(\log_a x)' = \frac{1}{x \ln a},$	$16.(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}},$
$7.(\mathrm{e}^x)' = \mathrm{e}^x,$	$17.(\tan^{-1} x)' = \frac{1}{1+x^2},$
$8.(\mathrm{a}^x)' = \mathrm{a}^x \ln \mathrm{a},$	$18.(\cot^{-1} x)' = -\frac{1}{1+x^2},$
$9.(\sin x)' = \cos x,$	$19.(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}},$
$10.(\cos x)' = -\sin x,$	$20.(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}.$

Differentiation formulas

Suppose c is a constant and both functions $f(x)$ and $g(x)$ are differentiable.

(a) $(cf(x))' = cf'(x),$

(b) $(f(x) + g(x))' = f'(x) + g'(x),$

(c) $(f(x) - g(x))' = f'(x) - g'(x),$

(d) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x),$

(e) $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.$

The Chain Rule

If the derivatives $g'(x)$ and $f'(g(x))$ both exist, and F(x)=f(g(x)), then F'(x) exist and $F'(x) = f'(g(x))g'(x)$.

Slopes of parametric curves

Suppose that a curve C is given by the parametric equations $x = x(t)$, $y = y(t)$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 4 Find $\frac{dy}{dx}$ for each function

(aa) $y = (\sin x)^x$.

The calculation of this derivative can be simplified by taking logarithms. This method is called *logarithmic differentiation*.

Steps in logarithmic differentiation

1. Take the logarithm of both sides of an equation.

$\ln y = \ln (\sin x)^x$

$\ln y = x \ln (\sin x)$

2. Differentiate implicitly with respect to x.

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln (\sin x))$

$\frac{y'}{y} = \ln (\sin x) + \frac{x}{\sin x}(\sin x)'$

$\frac{y'}{y} = \ln (\sin x) - \frac{x \cos x}{\sin x}$

3. Solve the resulting equation for y'.

$y' = y(\ln (\sin x) - \frac{x \cos x}{\sin x}) = (\sin x)^x(\ln (\sin x) - \frac{x \cos x}{\sin x})$

(ab) $y = \dfrac{\sqrt[5]{2x-1}(x^2-4)^2}{\sqrt[3]{1+3x}}$

$\ln y = \ln \dfrac{\sqrt[5]{2x-1}(x^2-4)^2}{\sqrt[3]{1+3x}} = \ln (\sqrt[5]{2x-1}) + \ln((x^2-4)^2) - \ln (\sqrt[3]{1+3x}) =$

$= \frac{1}{5} \ln (2x-1) + 2\ln(x-2) + 2\ln(x+2) - \frac{1}{3} \ln(1+3x)$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\frac{1}{5} \ln (2x-1) + 2\ln(x-2) + 2\ln(x+2) - \frac{1}{3} \ln(1+3x))$

$\frac{y'}{y} = \frac{1}{5(2x-1)}(2x-1)' + \frac{2}{x-2} + \frac{2}{x+2} - \frac{1}{3(1+3x)}(1+3x)' = \frac{2}{5(2x-1)} + \frac{2}{x-2} + \frac{2}{x+2} - \frac{1}{(1+3x)}$

$y' = y\left(\frac{2}{5(2x-1)} + \frac{2}{x-2} + \frac{2}{x+2} - \frac{1}{(1+3x)}\right) = \frac{\sqrt[5]{2x-1}(x^2-4)^2}{\sqrt[3]{1+3x}}\left(\frac{2}{5(2x-1)} + \frac{2}{x-2} + \frac{2}{x+2} - \frac{1}{(1+3x)}\right)$

(b) $y(t) = \sin^{-1} t$, $x(t) = \cos^{-1}(t^2)$.

$\dfrac{dy}{dt} = \dfrac{1}{\sqrt{1-t^2}}$,

$\dfrac{dx}{dt} = -\dfrac{2t}{\sqrt{1-t^4}}$,

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\dfrac{\frac{1}{\sqrt{1-t^2}}}{\frac{2t}{\sqrt{1-t^4}}} = -\dfrac{\sqrt{1+t^2}}{2t}$

(c) $2x^2 + 2xy + y^2 = x$.

We must use the method of *implicit differentiation*.

Differentiate both sides of the equation $2x^2 + 2xy + y^2 = x$ with respect to x

$\frac{d}{dx}(2x^2 + 2xy + y^2) = \frac{d}{dx}(x)$

Remembering that $y = y(x)$ and using the Chain Rule, we have

$4x + 2y + 2xy' + 2yy' = 1$

Now we solve the equation for y':

$2(x + y)y' = 1 - 4x - 2y$

$y' = \dfrac{1 - 4x - 2y}{2(x + y)}$

LINEAR AND QUADRATIC APPROXIMATIONS

The approximation $f(x) \approx f(a) + f'(a)(x - a)$ is called the *linear approximation* or *tangent line approximation* of f at a, and the function $L(x) = f(a) + f'(a)(x - a)$ is called the *linearization* of f at a.

The *quadratic approximation* of f near a is $f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.

Example 5 Find the quadratic approximation of $1/x$ for x near 4.

$f(x) \approx f(4) + f'(4)(x - 4) + \frac{f''(4)}{2}(x - 4)^2$

$f(4) = \frac{1}{4}$,

$f'(x) = -\frac{1}{x^2}$, $f'(4) = -\frac{1}{16}$,

$f''(x) = \frac{2}{x^3}$, $f''(4) = \frac{2}{64} = \frac{1}{32}$.

Thus,

$\frac{1}{x} \approx \frac{1}{4} - \frac{1}{16}(x - 4) + \frac{1}{64}(x - 4)^2$

L'HOSPITAL'S RULE

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ for points close to a (except, possibly a). Suppose that

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right| = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Indeterminate products If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} f(x)g(x) = |\infty \cdot 0| = \lim_{x\to a} \frac{f(x)}{1/g(x)} = \lim_{x\to a} \frac{g(x)}{1/f(x)} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right|$$

and now we can use L'Hospital's Rule.

Indeterminate differences If we have to find $\lim_{x\to a}(f(x) - g(x)) = |\infty - \infty|$ ($\lim_{x\to a} f(x) = \infty, \lim_{x\to a} g(x) = \infty$), then we have to convert the difference into a quotient (by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right|$ and we can use L'Hospital's Rule.

Indeterminate powers $\lim_{x\to a}[f(x)]^{g(x)} = |0^0 \text{ or } \infty^0 \text{ or } 1^\infty| = \lim_{x\to a} \mathrm{e}^{g(x)\ln f(x)} = \mathrm{e}^{\lim_{x\to a}[g(x)\ln f(x)]}$. Now let's find

$$\lim_{x\to a}[g(x)\ln f(x)] = |0 \cdot \infty| = \lim_{x\to a} \frac{\ln f(x)}{\frac{1}{g(x)}} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right| = b$$

Then

$$\lim_{x\to a}[f(x)]^{g(x)} = \mathrm{e}^b$$

Example 6 Evaluate each limit:

(a) $\lim_{x\to 0} \frac{\sin x + \sin 2x}{\sin 3x} = \left|\frac{0}{0}\right| = \lim_{x\to 0} \frac{\cos x + 2\cos 2x}{3\cos 3x} = \frac{1+2}{3} = 1$

(b) $\lim_{x\to 0}(\cot x - \csc x) = |\infty - \infty| = \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \left|\frac{0}{0}\right| = -\lim_{x\to 0} \frac{\sin x}{\cos x} = 0$

(c) $\lim_{x\to 0} x^{\sin x} = |0^0| = \lim_{x\to 0} \mathrm{e}^{\sin x \ln x} = \mathrm{e}^{\lim_{x\to 0}(\sin x \ln x)}$

Let's find

$$\lim_{x\to 0}(\sin x \ln x) = |0 \cdot \infty| = \lim_{x\to 0} \frac{\ln x}{\frac{1}{\sin x}} = \left|\frac{\infty}{\infty}\right| = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = -\lim_{x\to 0} \frac{\sin^2 x}{x\cos x} = -\lim_{x\to 0} \frac{\sin^2 x}{x} =$$

$$= -\lim_{x\to 0}(\sin x) = 0$$

Thus,

$$\lim_{x\to 0} x^{\sin x} = \mathrm{e}^0 = 1$$

MAXIMUM AND MINIMUM VALUES. DERIVATIVES AND THE SHAPES OF CURVES

A function f has an *absolute or global maximum* at c if $f(c) \geq f(x)$ for all x in the domain D of f. The number $f(c)$ is called the *maximum value* of f on D.

A function f has an *absolute minimum* at c if $f(c) \leq f(x)$ for all x in the domain D of f. The number $f(c)$ is called the *minimum value* of f on D.

The maximum and minimum values are called the *extreme value* of f.

A function f has an *local or relative maximum* at c if $f(c) \geq f(x)$ for all x near c.

A function f has an *local minimum* at c if $f(c) \leq f(x)$ for all x near c.

Fermat's Theorem If f has a local min or max at c, and if $f'(c)$ exists, then $f'(c) = 0$.

A *critical number* of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Increasing/decreasing test
(a) If $f'(x) > 0$ on an interval, then f is increasing on this interval.

(b) If $f'(x) > 0$ on an interval, then f is increasing on this interval.

The first derivative test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local max at c.

(b) If f' changes from negative to positive at c, then f has a local min at c.

(c) If f' does not change sign c, then f has a no local max or min at c.

A function is called *concave upward* (CU) on an interval I if f' is an increasing function on I. It is called *concave downward* (CD) on I if f' is an decreasing on I.

A point where a curve changes its direction of concavity is called an *inflection point*.

Concavity test

(a) If $f''(x) > 0$ on an interval, then f is CU on this interval.

(b) If $f''(x) < 0$ on an interval, then f is CD on this interval.

The second derivative test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.

Example 6 Determine when the function $f(x) = \mathrm{e}^{2x-x^2}$ is increasing, decreasing, concave up, concave down.

Let's find f' and f''.

$f'(x) = \mathrm{e}^{2x-x^2}(2x - x^2)' = (2 - 2x)\mathrm{e}^{2x-x^2} = 2(1 - x)\mathrm{e}^{2x-x^2}$.

$f'(x) = 0$ when $x = 1$. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$.

Thus, f is increasing for $x < 1$ and decreasing for $x > 1$ and f has a local max at $x = 1$.

$f''(x) = -2e^{2x-x^2} + 4(1-x)^2 e^{2x-x^2} = 2(2(1-x)^2 - 1)e^{2x-x^2}$.

$f''(x) = 0$ when $x = 1 + \frac{1}{\sqrt{2}}$ and $x = 1 - \frac{1}{\sqrt{2}}$

$f''(x) > 0$ for $x < 1 - \frac{1}{\sqrt{2}}$ or $x > 1 + \frac{1}{\sqrt{2}}$ and $f''(x) < 0$ for $1 - \frac{1}{\sqrt{2}} < x < 1 + \frac{1}{\sqrt{2}}$.

Thus, f is CU for $x < 1 - \frac{1}{\sqrt{2}}$ or $x > 1 + \frac{1}{\sqrt{2}}$ and CD for $1 - \frac{1}{\sqrt{2}} < x < 1 + \frac{1}{\sqrt{2}}$ and f has two inflection points $x = 1 + \frac{1}{\sqrt{2}}$ and $x = 1 - \frac{1}{\sqrt{2}}$.

APPLIED MAXIMUM AND MINIMUM PROBLEMS

Steps in solving applied max and min problems

1. Understand the problem.

2. Draw a diagram.

3. Introduce notation. Assign a symbol to the quantity that is to be minimized or maximized (let us call it Q). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols.

4. Express Q in terms of some of the other symbols from step 3.

5. If Q has been expressed as a function of more than one variable in step 4, use the given information to find relationships (in the form of equation) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus, Q will be given as a function of one variable.

6. Find the *absolute* maximum or minimum of Q.

Example 8 If 1200 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of that box.

Let A be the area of material, V be the volume of a box, h be the height of a box and a be the length of a box. Then $A = 1200$, $V = a^2 h$.

Since $A = a^2 + 4ah = 1200$ then $h = \frac{1200-a^2}{4a} = \frac{300}{a} - \frac{a}{4}$ and

$V = a^2 \left(\frac{300}{a} - \frac{a}{4}\right) = 300a - \frac{1}{4}a^3$.

Differentiating, we obtain

$\frac{dV}{da} = 300 - \frac{3}{4}a^2$,

so $\frac{dV}{da} = 300 - \frac{3}{4}a^2 = 0$ when $a^2 = 400$ or $a = 20$.

$h = \frac{300}{20} - \frac{20}{4} = 15 - 5 = 10$.

Then $V = 20^2(10) = 4000\text{cm}^3$.

ANTIDERIVATIVE. DEFINITE AND INDEFINITE INTEGRALS

Function $F(x)$ is called an *antiderivative* of $f(x)$ if $F'(x) = f(x)$.

Then $\int f(x)dx = F(x) + C$ (indefinite integral=antiderivative)

Table of indefinite integrals

1. $\int adx = x + C$, a is a constant,
2. $\int xdx = \frac{x^2}{2} + C$,
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$,
4. $\int \frac{1}{x}dx = \ln|x| + C$,
5. $\int \mathrm{e}^x dx = \mathrm{e}^x + C$,
6. $\int \mathrm{a}^x dx = \frac{\mathrm{a}^x}{\ln \mathrm{a}}$,
7. $\int \sin xdx = -\cos x + C$,
8. $\int \cos xdx = \sin x + C$,
9. $\int \tan xdx = -\ln|\cos x| + C$,
10. $\int \cot xdx = \ln|\sin x| + C$,
11. $\int \sec^2 xdx = \tan x + C$,
12. $\int \csc^2 xdx = -\cot x + C$,
13. $\int \sec x \tan xdx = \sec x + C$,
14. $\int \csc x \cot x = -\csc x + C$,
15. $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$,
16. $\int \frac{1}{1+x^2}dx = \tan^{-1} + C$.

Properties of indefinite integrals

1. $\int af(x)dx = a\int f(x)dx$,
2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$,
3. $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$.

A vector function $\vec{R}(t) =< X(t), Y(t) >$ is called an *antiderivative* of $\vec{r}(t) =< x(t), y(t) >$ if $\vec{R}'(t) = \vec{r}(t)$: that is, $X'(t) = x(t)$ and $Y'(t) = y(t)$.

The fundamental theorem of calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is an antiderivative of f.

Properties of the definite integral

1. $\int_a^b cdx = c(b - a)$, where c is a constant.
2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is a constant.
3. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
4. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$.

6. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

7. If $f(x) \geq 0$ for $a < x < b$, then $\int_a^b f(x)dx \geq 0$.

8. If $f(x) \geq g(x)$ for $a < x < b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

9. If $m \leq f(x) \leq M$ for $a < x < b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

10. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Example 9

(a) Find the area under the curve $y = \sqrt{x}$ above the x-axis between 0 and 4.

area=$\int_0^4 \sqrt{x}dx = \int_0^4 x^{\frac{1}{2}}dx = \left.\frac{x^{1/2+1}}{\frac{1}{2}+1}\right|_0^4 = \left.\frac{2x^{\frac{3}{2}}}{3}\right|_0^4 = \frac{2(4^{\frac{3}{2}-0})}{3} = \frac{16}{3}$.

(b) A particle moves in a straight line and has acceleration given by $a(t) = t^2 - t$. Its initial velocity is $v(0) = 2$ cm/s and its initial diplacement is $s(0) = 1$ cm. Find the position function $s(t)$.

Since $v'(t) = a(t)$, then

$v(t) = \int(t^2 - t)dt = \frac{t^3}{3} - \frac{t^2}{2} + C$.

$v(0) = C = 2$ and

$v(t) = \frac{t^3}{3} - \frac{t^2}{2} + 2$.

Since $s'(t) = v(t)$, antiderivatives gives

$s(t) = \int(\frac{t^3}{3} - \frac{t^2}{2} + 2)dt = \frac{t^4}{12} - \frac{t^3}{6} + 2t + C$.

$s(0) = C = 1$ and

$s(t) = \frac{t^4}{12} - \frac{t^3}{6} + 2t + 1$.

(c) Find the vector function $\vec{r}(t)$ that gives the position of a particle at time t having the acceleration $\vec{a}(t) = 2t\vec{\imath} + \vec{\jmath}$, initial velocity $\vec{v}(0) = \vec{\imath} - \vec{\jmath}$, and initial position $(1, 0)$.

$\vec{a}(t) = 2t\vec{\imath} + \vec{\jmath} =< 2t, 1 >$.

Since $\vec{v}(t)$ is an antiderivative of $\vec{a}(t)$, we have

$\vec{v}(t) =< \int 2tdt, \int 1dt >=< t^2 + C_1, t + C_2 >$,

$\vec{v}(0) =< C_1, C_2 >=< 1, -1 >$, so $C_1 = 1$ and $C_2 = -1$ and

$\vec{v}(t) =< t^2 + 1, t - 1 >$.

Since $\vec{r}(t)$ is an antiderivative of $\vec{v}(t)$, we have

$\vec{r}(t) =< \int(t^2+1)dt, \int(t-1)dt >=< \frac{t^3}{3}+t+C_3, \frac{t^2}{2}-t+C_4 >$,

$\vec{r}(0) =< C_3, C_4 >=< 1, 0 >$, so $C_3 = 1$ and $C_4 = 0$ and

$\vec{r}(t) =< \frac{t^3}{3}+t+1, \frac{t^2}{2}-t >$.

THE SUBSTITUTION RULE

The substitution rule for indefinite integrals If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$\int f(g(x))g'(x)dx = \int f(u)du.$

The substitution rule for definite integrals If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of g, then

$\int\limits_a^b f(g(x))g'(x)dx = \int\limits_{g(a)}^{g(b)} f(u)du.$

Example 10 Evaluate each integral

(a) $\int t^2\cos(1-t^3)dt = \left| \begin{array}{c} u = 1-t^3 \\ du = -3t^2dt \end{array} \right| = -\frac{1}{3}\int \cos u du = \sin u + C = \sin(1-t^3) + C.$

(b) $\int\limits_1^2 x2^{x^2}dx = \left| \begin{array}{cc} t = x^2 & 1 \to 1 \\ dt = 2xdx & 2 \to 4 \end{array} \right| = \frac{1}{2}\int\limits_1^4 2^t dt = \frac{1}{2}\frac{2^t}{\ln 2}\Big|_1^4 = \frac{2^4-2^1}{2\ln 2} = \frac{7}{\ln 2}.$