# Math 151, Spring 2008 <br> Review Before Final 

04/30/2008

## Table of derivatives

$$
\begin{array}{ll}
\text { 1. }(C)^{\prime}=0, C \text { is a constant, } & \text { 11. }(\tan x)^{\prime}=\sec ^{2} x, \\
\text { 2. }(x)^{\prime}=1, & 12 .(\cot x)^{\prime}=-\csc ^{2} x, \\
\text { 3. }\left(x^{2}\right)^{\prime}=2 x, & \text { 13. }(\sec x)^{\prime}=\sec x \tan x, \\
\text { 4. }\left(x^{n}\right)^{\prime}=n x^{n-1}, & \text { 14. }(\csc x)^{\prime}=-\csc x \cot x, \\
\text { 5. }(\ln x)^{\prime}=\frac{1}{x}, & 15 .\left(\sin ^{-1} x\right)^{\prime}=\frac{1}{\sqrt{1-x^{2}}}, \\
\text { 6. }\left(\log _{a} x\right)^{\prime}=\frac{1}{x \ln a}, & 16 .\left(\cos ^{-1} x\right)^{\prime}=-\frac{1}{\sqrt{1-x^{2}}}, \\
\text { 7. }\left(\mathrm{e}^{x}\right)^{\prime}=\mathrm{e}^{x}, & \text { 17. }\left(\tan ^{-1} x\right)^{\prime}=\frac{1}{1+x^{2}}, \\
\text { 8. }\left(\mathrm{a}^{x}\right)^{\prime}=\mathrm{a}^{x} \ln a, & \text { 18. }\left(\cot ^{-1} x\right)^{\prime}=-\frac{1}{1+x^{2}}, \\
\text { 9. }(\sin x)^{\prime}=\cos x, & \text { 19. }\left(\sec ^{-1} x\right)^{\prime}=\frac{1}{x \sqrt{x^{2}-1}}, \\
\text { 10. }(\cos x)^{\prime}=-\sin x, & \text { 20. }\left(\csc ^{-1} x\right)^{\prime}=-\frac{1}{x \sqrt{x^{2}-1}} .
\end{array}
$$

## Table of indefinite integrals

1. $\int a d x=x+C, a$ is a constant, 9. $\int \tan x d x=-\ln |\cos x|+C$,
2. $\int x d x=\frac{x^{2}}{2}+C$, 10. $\int \cot x d x=\ln |\sin x|+C$,
3. $\int x^{n} d x=\frac{x^{n+1}}{n+1}+C, n \neq-1$,
4. $\int \sec ^{2} x d x=\tan x+C$,
5. $\int \frac{1}{x} d x=\ln |x|+C$,
6. $\int \mathrm{e}^{x} d x=\mathrm{e}^{x}+C$,
7. $\int a^{x} d x=\frac{a^{x}}{\ln \mathrm{a}}$,
8. $\int \sin x d x=-\cos x+C$,
9. $\int \cos x d x=\sin x+C$,
10. $\int \sec x \tan x d x=\sec x+C$, 14. $\int \csc x \cot x=-\csc x+C$,
11. $\int \frac{1}{\sqrt{1-x^{2}}} d x=\sin ^{-1} x+C$,
12. $\int \frac{1}{1+x^{2}} d x=\tan ^{-1}+C$.

## Vectors. The dot product

Let c be a scalar and $\vec{a}=<a_{1}, a_{2}>=a_{1} \vec{\imath}+a_{2} \vec{\jmath}$ and $\vec{b}=<b_{1}, b_{2}>=b_{1} \vec{\imath}+b_{2} \vec{\jmath}$ be vectors. Then

$$
c \vec{a}=<c a_{1}, c a_{2}>
$$

$$
\vec{a} \pm \vec{b}=<a_{1} \pm b_{1}, a_{2} \pm b_{2}>,
$$

the magnitude of $\vec{a}$ is

$$
|\vec{a}|=\sqrt{a_{1}^{2}+a_{2}^{2}},
$$

vector $\vec{u}=\frac{1}{|\vec{a}|} \vec{a}=\left\langle\frac{a_{1}}{|\overrightarrow{\mid}|}, \frac{a_{2}}{|\vec{a}|}\right\rangle$ is a unit vector that has the same direction as $\vec{a}$, the dot product of vectors $\vec{a}$ and $\vec{b}$ :
$\vec{a} \cdot \vec{b}=|\vec{a}||\vec{b}| \cos \theta=a_{1} b_{1}+a_{2} b_{2}$
the scalar projection of $\vec{b}$ onto $\vec{a}$ is a number $\operatorname{comp}_{\vec{a}} \vec{b}=\left|\frac{\vec{a}}{\vec{a} \vec{b} \mid}\right|$, the vector projection of $\vec{b}$ onto $\vec{a}$ is a vector $\operatorname{proj}_{\vec{a}} \vec{b}=\frac{\vec{b} \cdot \vec{b}}{|\vec{a}|^{2}} \vec{a}$. Given the points $A\left(x_{1}, y_{1}\right)$ and $B\left(x_{2}, y_{2}\right)$, then $\overrightarrow{A B}=<x_{2}-x_{1}, y_{2}-y_{1}>$.

Definition Given the nonzero vector $\vec{a}=<a_{1}, a_{2}>$, the orthogonal complement of $\vec{a}$ is the vector $\vec{a}^{\perp}=<-a_{2}, a_{1}>$.


The distance from the point $P$ to the line $L P Q=\operatorname{comp}_{\overrightarrow{A B}}^{\perp} \overrightarrow{A P}$.

Example 1. Find the angle between vectors $\vec{a}=<3,4>$ and $\vec{b}=<-1,2>$.

Example 1. Find the angle between vectors $\vec{a}=<3,4>$ and $\vec{b}=\langle-1,2\rangle$.

Example 2. Find the distance between the line $y=x$ and the point $(0,1)$.

## Vector equation of a line

A line $L$ is determined by a point $P_{0}$ on $L$ and a direction. Let $\vec{v}$ be a vector parallel to line $L$. Let $P$ be be an arbitrary point on $L$ and let $\overrightarrow{r_{0}}$ and $\vec{r}$ be the position vectors of $P$ and $P_{0}$. Then the vector equation of line $L$ is
$\vec{r}(t)=\overrightarrow{r_{0}}+t \vec{v}$
If $\vec{r}=<x(t), y(t)>, \vec{v}=<a, b>$ and $P\left(x_{0}, y_{0}\right)$ then parametric
equations of the line $L$ are
$x(t)=x_{0}+a t, y(t)=y_{0}+b t$.
Example 3. Find a vector equation and parametric equations to the line that passes through the points $M_{1}(1,-2)$ and $M_{2}(2,0)$.

## Continuity. Vertical and horizontal asymptotes

Function $f$ is continuous at a number a if $\lim _{x \rightarrow a} f(x)=f(a)$.
If $f$ is not continuous at $a$, then $f$ has discontinuity at a.
If $\lim _{x \rightarrow a^{+}} f(x) \neq \lim _{x \rightarrow a^{-}} f(x)$, then $f$ has a jump discontinuity at $a$, if either $\lim _{x \rightarrow a^{+}} f(x)=\infty$ or $\lim _{x \rightarrow a^{-}} f(x)=\infty$, then $f$ has an infinity discontinuity at $a$ and we say line $x=a$ is a vertical asymptote of the curve $y=f(x)$.
and if $\lim _{x \rightarrow a^{+}} f(x)=\lim _{x \rightarrow a^{-}} f(x) \neq f(a)$, then $f$ has a removable discontinuity at $a$.

If $\lim _{x \rightarrow \infty} f(x)=b$, then we say line $y=b$ is horizontal asymptote of the curve $y=f(x)$.
Example 4. Find all vertical asymptotes of the curve $y=\frac{x+7}{x^{2}-49}$

## Derivatives

## Differentiation formulas

Suppose $c$ is a constant and both functions $f(x)$ and $g(x)$ are differentiable.
(a) $(c f(x))^{\prime}=c f^{\prime}(x)$,
(b) $(f(x)+g(x))^{\prime}=f^{\prime}(x)+g^{\prime}(x)$,
(c) $(f(x)-g(x))^{\prime}=f^{\prime}(x)-g^{\prime}(x)$,
(d) $(f(x) g(x))^{\prime}=f^{\prime}(x) g(x)+f(x) g^{\prime}(x)$,
(e) $\left(\frac{f(x)}{g(x)}\right)^{\prime}=\frac{f^{\prime}(x) g(x)-f(x) g^{\prime}(x)}{g^{2}(x)}$.

## The Chain Rule

If the derivatives $g^{\prime}(x)$ and $f^{\prime}(g(x))$ both exist, and $F(x)=f(g(x))$, then $F^{\prime}(x)$ exist and $F^{\prime}(x)=f^{\prime}(g(x)) g^{\prime}(x)$.

## Slopes of parametric curves

Suppose that a curve $C$ is given by the parametric equations
$x=x(t), y=y(t)$, then $\frac{d y}{d x}=\frac{\frac{d y}{d t}}{\frac{d x}{d t}}$

Example 5. A curve is given by the parametric equations $x(t)=t^{3}-1+\ln t, y(t)=t^{2}$. Find the slope of the tangent line at $t=1$.

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Example 6. Find the equation of the tangent line to the curve $x^{3}+y^{3}=6 x y$ at the point $(3,3)$.

## Steps in logarithmic differentiation

1. Take the logarithm of both sides of an equation.
2. Differentiate implicitly with respect to $x$.
3. Solve the resulting equation for $y^{\prime}$.

## Higher derivatives

$$
f^{\prime \prime}(x)=\left[f^{\prime}(x)\right]^{\prime}, f^{\prime \prime \prime}(x)=\left[f^{\prime \prime}(x)\right]^{\prime}, \ldots, f^{(n)}(x)=\left[f^{(n-1)}(x)\right]^{\prime}
$$

Example 7. Find $(\sin (2 x))^{(2008)}$, i.e. the 2008 -th derivative of $\sin (2 x)$.

## Linear and quadratic approximations

The approximation $f(x) \approx f(a)+f^{\prime}(a)(x-a)$ is called the linear approximation or tangent line approximation of $f$ at $a$, and the function $L(x)=f(a)+f^{\prime}(a)(x-a)$ is called the linearization of $f$ at .
The quadratic approximation of $f$ near $a$ is
$f(x) \approx f(a)+f^{\prime}(a)(x-a)+\frac{f^{\prime \prime}(a)}{2}(x-a)^{2}$
Example 8. Find the quadratic approximation to the function $f(x)=\sqrt{1+x}$ near 0 .

## Maximum and minimum values. Derivatives and the shapes of curves

A function $f$ has an absolute or global maximum at $c$ if $f(c) \geq f(x)$ for all $x$ in the domain $D$ of $f$. The number $f(c)$ is called the maximum value of $f$ on $D$.

A function $f$ has an absolute minimum at $c$ if $f(c) \leq f(x)$ for all $x$ in the domain $D$ of $f$. The number $f(c)$ is called the minimum value of $f$ on $D$.

The maximum and minimum values are called the extreme value of $f$.

A function $f$ has an local or relative maximum at $c$ if $f(c) \geq f(x)$ for all $x$ near $c$.

A function $f$ has an local minimum at $c$ if $f(c) \leq f(x)$ for all $x$ near $c$.

## Fermat's Theorem

If $f$ has a local min or max at $c$, and if $f^{\prime}(c)$ exists, then
$f^{\prime}(c)=0$.
A critical number of a function $f$ is a number $c$ in the domain of $f$ such that either $f^{\prime}(c)=0$ or $f^{\prime}(c)$ does not exist.

## Increasing/decreasing test

(a) If $f^{\prime}(x)>0$ on an interval, then $f$ is increasing on this interval.
(b) If $f^{\prime}(x)>0$ on an interval, then $f$ is increasing on this interval.

The first derivative test Suppose that $c$ is a critical number of a continuous function $f$.
(a) If $f^{\prime}$ changes from positive to negative at $c$, then $f$ has a local $\max$ at $c$.
(b) If $f^{\prime}$ changes from negative to positive at $c$, then $f$ has a local $\min$ at $c$.
(c) If $f^{\prime}$ does not change sign $c$, then $f$ has a no local max or min at $c$.

Example 9. Find and classify all critical points of the function $f(x)=4 x^{3}-9 x^{2}-12 x+3$.

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The closed interval method To find the absolute maximum and minimum values of a continuous function $f$ on a closed interval $[a, b]$ :
(a) Find the values of $f$ at the critical numbers of $f$ in $(a, b)$
(b) Find $f(a)$ and $f(b)$
(c) The largest number of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

A function is called concave upward (CU) on an interval $/$ if $f^{\prime}$ is an increasing function on $I$. It is called concave downward (CD) on $l$ if $f^{\prime}$ is an decreasing on $l$.

A point where a curve changes its direction of concavity is called an inflection point.

## Concavity test

(a) If $f^{\prime \prime}(x)>0$ on an interval, then $f$ is CU on this interval.
(b) If $f^{\prime \prime}(x)<0$ on an interval, then $f$ is CD on this interval.

The second derivative test Suppose $f^{\prime \prime}$ is continuous near $c$. (a) If $f^{\prime}(c)=0$ and $f^{\prime \prime}(c)>0$, then $f$ has a local min at $c$.
(b) If $f^{\prime}(c)=0$ and $f^{\prime \prime}(c)<0$, then $f$ has a local max at $c$.

Example 10. Find the intervals of the concavity and the inflection point(s) of the function $f(x)=\left(1+x^{2}\right) \mathrm{e}^{x}$.

## Applied maximum and minimum problems

Steps in solving applied max and min problems

1. Understand the problem.
2. Draw a diagram.
3. Introduce notation. Assign a symbol to the quantity that is to be minimized or maximized (let us call it $Q$ ). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols.
4. Express $Q$ in terms of some of the other symbols from step 3.
5. If $Q$ has been expressed as a function of more than one variable in step 4, use the given information to find relationships (in the form of equation) among these variables. Then use these equations to eliminate all but one of the variables in the expression for $Q$. Thus, $Q$ will be given as a function of one variable.
6 . Find the absolute maximum or minimum of $Q$.

Example 11. A farmer with 1000 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

## Indeterminate forms and L'Hospitale's rule

L'Hospital's Rule Suppose $f$ and $g$ are differentiable and $g^{\prime}(x) \neq 0$ for points close to a (except, possibly a). Suppose that $\lim _{x \rightarrow a} f(x)=0$ and $\lim _{x \rightarrow a} g(x)=0$ or $\lim _{x \rightarrow a} f(x)=\infty$ and $\lim _{x \rightarrow a} g(x)=\infty$. Then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\left\lvert\, \frac{0}{0}\right. \text { or } \frac{\infty}{\infty} \left\lvert\,=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)}\right.
$$

Indeterminate products If $\lim _{x \rightarrow a} f(x)=\infty$ and $\lim _{x \rightarrow a} g(x)=0$ or $\lim _{x \rightarrow a} f(x)=0$ and $\lim _{x \rightarrow a} g(x)=\infty$, then
$\lim _{x \rightarrow a} f(x) g(x)=|\infty \cdot 0|=\lim _{x \rightarrow a} \frac{f(x)}{1 / g(x)}=\lim _{x \rightarrow a} \frac{g(x)}{1 / f(x)}=\left\lvert\, \frac{0}{0}\right.$ or $\left.\frac{\infty}{\infty} \right\rvert\,$
and now we can use L'Hospital's Rule.

Indeterminate differences If we have to find $\lim _{x \rightarrow a}(f(x)-g(x))=|\infty-\infty|\left(\lim _{x \rightarrow a} f(x)=\infty, \lim _{x \rightarrow a} g(x)=\infty\right)$, then we have to convert the difference into a quotient (by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\left\lvert\, \frac{0}{0}\right.$ or $\left.\frac{\infty}{\infty} \right\rvert\,$ and we can use L'Hospital's Rule.

Indeterminate powers $\lim _{x \rightarrow a}[f(x)]^{g(x)}=\mid 0^{0}$ or $\infty^{0}$ or $1^{\infty} \mid=$
$\lim _{x \rightarrow a} \mathrm{e}^{g(x) \ln f(x)}=\mathrm{e}^{\lim _{x \rightarrow a}[g(x) \ln f(x)]}$. Now let's find
$\lim _{x \rightarrow a}[g(x) \ln f(x)]=|0 \cdot \infty|=\lim _{x \rightarrow a} \frac{\ln f(x)}{\frac{1}{g(x)}}=\left\lvert\, \frac{0}{0}\right.$ or $\left.\frac{\infty}{\infty} \right\rvert\,=b$
Then
$\lim _{x \rightarrow a}[f(x)]^{g(x)}=\mathrm{e}^{b}$

Example 12. Find each limit:
(a) $\lim _{x \rightarrow 0} \frac{\tan x-\sin x}{x-\sin x}$

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(b) $\lim _{x \rightarrow \infty} x^{3} e^{-x^{2}}$

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(b) $\lim _{x \rightarrow \infty} x^{3} \mathrm{e}^{-x^{2}}$
(c) $\lim _{x \rightarrow 0}(1-2 x)^{1 / x}$

## Inverse functions

Definition A function $f$ with domain $A$ is called one-to-one function if $f\left(x_{1}\right) \neq f\left(x_{2}\right)$ whenever $x_{1} \neq x_{2}$.

Horizontal line test A function is one-to-one if and only if no horizontal line intersects its graph more that once.

Definition Let $f$ be one-to-one function with domain $A$ and range $B$. Then its inverse function $f^{-1}$ has domain $B$ and range $A$ and is defined by $f^{-1}(y)=x \Longleftrightarrow f(x)=y$ for any $y$ in $B$.
domain of $f^{-1}=$ range of $f$
range of $f^{-1}=$ domain of $f$
Let $f$ be one-to-one function with domain $A$ and range $B$. If $f(a)=b$, then $f^{-1}(b)=a$. Cancellation equations
$f^{-1}(f(x))=x$ for every $x \in A$
$f\left(f^{-1}(x)\right)=x$ for every $x \in B$

How to find the inverse function of a one-to-one function $f$

1. Write $y=f(x)$
2. Solve this equation for $x$ in terms of $y$.
3. Interchange $x$ and $y$. The resulting equation is $y=f^{-1}(x)$.

The graph of $f^{-1}$ is obtained by the reflecting the graph $f$ about the line $y=x$.

Theorem If $f$ is a one-to-one differentiable function with inverse function $g=f^{-1}$ and $f^{\prime}(g(a)) \neq 0$, then the inverse function is
differentiable at $a$ and $g^{\prime}(a)=\frac{1}{f^{\prime}(g(a))}$
Example 13. If $f(x)=x^{7}+3 x+1$ and $g=f^{-1}$, find $g^{\prime}(1)$.

## Exponential function.

An exponential function is a function of the form $f(x)=a^{x}$ where $a$ is a positive constant.

Theorem If $a>0$ and $a \neq 1$, then $f(x)=a^{x}$ is continuous function with domain $(-\infty, \infty)$ and range $(0, \infty)$.

If $0<a<1, f(x)=a^{x}$ is decreasing function
if $a>1, f(x)=a^{x}$ is increasing function
If $a, b>0$ and $x, y$ are reals, then

$$
\text { 1. } a^{x+y}=a^{x} a^{y} \quad 2 . a^{x-y}=\frac{a^{x}}{a^{y}} \quad 3 .\left(a^{x}\right)^{y}=a^{x y} \quad \text { 4. }(a b)^{x}=a^{x} b^{x}
$$

If $0<a<1, \lim _{x \rightarrow-\infty} a^{x}=\infty, \lim _{x \rightarrow \infty} a^{x}=0$
If $a>1, \lim _{x \rightarrow-\infty} a^{x}=0, \lim _{x \rightarrow \infty} a^{x}=\infty$
$\lim _{x \rightarrow 0} \frac{\mathrm{e}^{x}-1}{x}=1, \mathrm{e} \approx 2.7283>1$

## Logarithmic function

$\log _{a} x=y \Longleftrightarrow a^{y}=x$
The cancellation equations
$\log _{a} a^{x}=x \quad a^{\log _{a} x}=x$
Theorem Function $f(x)=\log _{a} x$ is one-to-one continuous function with domain $(0, \infty)$ and range $(-\infty, \infty)$.
If $a>1$, then
$f(x)=\log _{a} x$ is increasing function, $\lim _{x \rightarrow \infty} \log _{a} x=\infty$

| $\lim _{x \rightarrow 0^{+}} \log _{a} x=-\infty$ |
| :--- |
| if $0<a<1$, then |

$f(x)=\log _{a} x$ is decreasing function, $\lim _{x \rightarrow \infty} \log _{a} x=-\infty$

$$
\lim _{x \rightarrow 0^{+}} \log _{a} x=\infty
$$

$$
\log _{\mathrm{e}} x=\ln x
$$

If $x, y>0$ and $k$ is a constant, then

1. $\log _{a} x y=\log _{a} x+\log _{a} y$
2. $\log _{a} \frac{x}{y}=\log _{a} x-\log _{a} y$
3. $\log _{a} x^{k}=k \log _{a} x$
4. $\log _{a^{k}} x=\frac{1}{k} \log _{a} x$

$$
\log _{\frac{1}{a}} x=-\log _{a} x
$$

5. $\log _{a} a=1$
6. $\log _{a} 1=0$
7. $\log _{a} x=\frac{\log _{b} x}{\log _{b} a}$

Example 14. $e^{\ln 6-\frac{3}{4} \ln 16}=$

## Inverse trigonometric functions

Inverse sine function

$$
\arcsin x=\sin ^{-1} x=y \Leftrightarrow \sin y=x
$$

DOMAIN $\quad-1 \leq x \leq 1 \quad$ RANGE $\quad-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
CANCELLATION EQUATIONS
$\sin ^{-1}(\sin x)=x \quad$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
$\sin \left(\sin ^{-1} x\right)=x$ for $-1 \leq x \leq 1$

## Inverse cosine function

$\arccos x=\cos ^{-1} x=y \quad \Leftrightarrow \quad \cos y=x$
DOMAIN $\quad-1 \leq x \leq 1 \quad$ RANGE $\quad 0 \leq y \leq \pi$

CANCELLATION EQUATIONS

| $\cos ^{-1}(\cos x)=x$ | for $0 \leq x \leq \pi$ |
| :--- | :--- |
| $\cos \left(\cos ^{-1} x\right)=x$ | for $-1 \leq x \leq 1$ |

## Inverse tangent function

$\arctan x=\tan ^{-1} x=y \Leftrightarrow \tan y=x$
DOMAIN $\quad-\infty \leq x \leq \infty \quad$ RANGE $\quad-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
CANCELLATION EQUATIONS

| $\tan ^{-1}(\tan x)=x$ | for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ |
| :--- | :--- |
| $\tan \left(\tan ^{-1} x\right)=x$ | for $-\infty \leq x \leq \infty$ |

$\lim _{x \rightarrow-\infty} \tan ^{-1} x=-\frac{\pi}{2}$
$\lim _{x \rightarrow \infty} \tan ^{-1} x=\frac{\pi}{2}$
Example 15. $\cos \left[2 \sin ^{-1}\left(\frac{5}{13}\right)\right]=$

## Sigma notation

Definition If $a_{m}, a_{m+1}, a_{m+2}, \ldots, a_{n}$ are real numbers and $m$ and $n$ are integers such that $m \leq n$, then

$$
a_{m}+a_{m+1}+a_{m+2}+\ldots+a_{n}=\sum_{i=m}^{n} a_{i}
$$

Theorem If $c$ is any constant (this means that $c$ does not depend on $i$ ), then
(a) $\sum_{i=1}^{n} c a_{i}$
$\sum_{i=1}^{n} 1=n$
(b) $\sum_{i=1}^{n}\left(a_{i} \pm b_{i}\right)=\sum_{i=1}^{n} a_{i} \pm \sum_{i=1}^{n} b_{i}$
$\sum_{i=1}^{n} i^{3}=\left[\frac{n(n+1)}{2}\right]^{2}$

$$
\sum_{i=1}^{n} i^{4}=\frac{n(n+1)(2 n+1)\left(3 n^{2}+3 n-1\right)}{30}
$$

Example 16. Compute $\sum_{i=1}^{20} i(i+2)$.

## Antiderivative. Definite and indefinite integrals

Function $F(x)$ is called an antiderivative of $f(x)$ if $F^{\prime}(x)=f(x)$.
Then $\int f(x) d x=F(x)+C$ (indefinite integral=antiderivative)

## Properties of indefinite integrals

1. $\int a f(x) d x=a \int f(x) d x$,
2. $\int[f(x)+g(x)] d x=\int f(x) d x+\int g(x) d x$,
3. $\int[f(x)-g(x)] d x=\int f(x) d x-\int g(x) d x$.

A vector function $\vec{R}(t)=<X(t), Y(t)>$ is called an
antiderivative of $\vec{r}(t)=<x(t), y(t)>$ if $\vec{R}^{\prime}(t)=\vec{r}(t)$ : that is, $X^{\prime}(t)=x(t)$ and $Y^{\prime}(t)=y(t)$.

The fundamental theorem of calculus Suppose $f$ is continuous on $[a, b]$.

1. If $g(x)=\int_{a}^{x} f(t) d t$, then $g^{\prime}(x)=f(x)$.
2. $\int_{a}^{b} f(x) d x=F(b)-F(a)$, where $F$ is an antiderivative of $f$.

## Properties of the definite integral

1. $\int_{a}^{b} c d x=c(b-a)$, where $c$ is a constant.
2. $\int_{a}^{b} c f(x) d x=c \int_{a}^{b} f(x) d x$, where $c$ is a constant.
3. $\int_{a}^{b}[f(x) \pm g(x)] d x=\int_{a}^{b} f(x) d x \pm \int_{a}^{b} g(x) d x$.
4. $\int_{a}^{b} f(x) d x=\int_{a}^{c} f(x) d x+\int_{c}^{b} f(x) d x$, where $a<c<b$.
5. $\int_{a}^{b} f(x) d x=-\int_{b}^{a} f(x) d x$.
6. If $f(x) \geq 0$ for $a<x<b$, then $\int_{a}^{b} f(x) d x \geq 0$.
7. If $f(x) \geq g(x)$ for $a<x<b$, then $\int_{a}^{b} f(x) d x \geq \int_{a}^{b} g(x) d x$.
8. If $m \leq f(x) \leq M$ for $a<x<b$, then
$m(b-a) \leq \int_{a}^{b} f(x) d x \leq M(b-a)$.
9. $\left|\int_{a}^{b} f(x) d x\right| \leq \int_{a}^{b}|f(x)| d x$

Example 17. Find the area bounded by the graph of the function $y=x^{3}$, the $x$-axis, and the line $x=1$.
6. If $f(x) \geq 0$ for $a<x<b$, then $\int_{a}^{b} f(x) d x \geq 0$.
7. If $f(x) \geq g(x)$ for $a<x<b$, then $\int_{a}^{b} f(x) d x \geq \int_{a}^{b} g(x) d x$.
8. If $m \leq f(x) \leq M$ for $a<x<b$, then
$m(b-a) \leq \int_{a}^{b} f(x) d x \leq M(b-a)$.
9. $\left|\int_{a}^{b} f(x) d x\right| \leq \int_{a}^{b}|f(x)| d x$

Example 17. Find the area bounded by the graph of the function $y=x^{3}$, the $x$-axis, and the line $x=1$.
Example 18. A particle moves along a straight line and has acceleration given by $a(t)=t^{2}-t$. Its initial velocity is $v(0)=2$ $\mathrm{cm} / \mathrm{s}$ and its initial displacement is $s(0)=1 \mathrm{~cm}$. Find the position function $s(t)$.

Example 19. Find the vector function $\vec{r}(t)$ that gives the position of a particle at time $t$ having the acceleration $\vec{a}(t)=2 t \vec{\imath}+\vec{\jmath}$, initial velocity $\vec{v}(0)=\vec{\imath}-\vec{\jmath}$, and initial position ( 1,0 ).

## The Substitution Rule

The substitution rule for indefinite integrals If $u=g(x)$ is a differentiable function whose range is an interval $/$ and $f$ is
continuous on 1 , then
$\int f(g(x)) g^{\prime}(x) d x=\int f(u) d u$
The substitution rule for definite integrals If $g^{\prime}(x)$ is continuous on $[a, b]$ and $f$ is continuous on the range of $g$, then $\int_{a}^{b} f(g(x)) g^{\prime}(x) d x=\int_{g(a)}^{g(b)} f(u) d u$

Example 20. Evaluate the integral $\int_{e}^{e^{2}} \frac{1}{t \ln ^{2} t} d t$.

