Math 152, Fall 2008

Lecture 4.

09/04/2008

Chapter 7. **Applications of integration**
Section 7.2 **Volume**

Definition of volume Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is an integrable function, then the **volume** of S is

$$\boxed{V = \int_a^b A(x)dx}$$

Let S be the solid obtained by revolving the plane region $\mathcal{R}$ bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the x-axis. We have the following **formula for a volume of revolution:**

$$\boxed{V = \pi \int_a^b [f(x)]^2 dx}$$

The region bounded by the curves $x = g(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the y-axis, then the corresponding volume of revolution is

$$\boxed{V = \pi \int_c^d [g(y)]^2 dy}$$

Let S be the solid generated when the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$, $x = a$, and $x = b$ (where $f_1(x) \geq f_2(x)$ for all x in $[a, b]$) is rotated about the x-axis. Then the volume of S is

$$\boxed{V = \pi \int_a^b \{[f_1(x)]^2 - [f_2(x)]^2\} dx}$$

The region bounded by the curves $x = g_1(y)$, $x = g_2(x)$, $y = c$, and $y = d$ (where $g_1(x) \geq g_2(x)$ for all y in $[c, d]$) is rotated about the y-axis, then the corresponding volume of revolution is

$$\boxed{V = \pi \int_c^d \{[g_1(y)]^2 - [g_2(y)]^2\} dy}$$

Example 1. Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 1$ about $y = 2$.

Example 2. Find the volume of a frustum of a pyramid with square base of side b, square top of side a, and height h.

Section 7.3 **Volumes by cylindrical shells**

Lets find the volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h (see Fig.1).

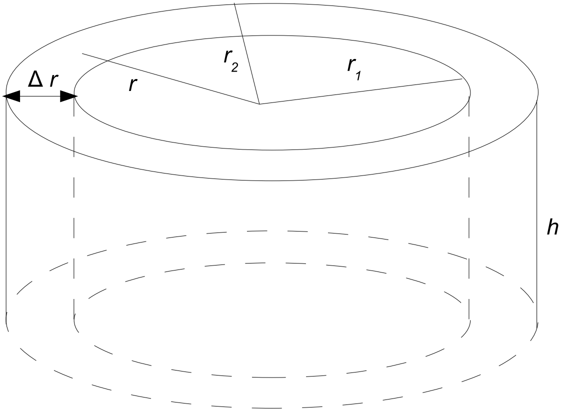

Fig.1

V can be calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1 = \pi h(r_2^2 - r_1^2) = 2\pi h \frac{r_2 + r_1}{2}(r_2 - r_1)$$

Let $\Delta r = r_2 - r_1$, $r = (r_2 + r_1)/2$, then the volume of a cylindrical shell is $\boxed{V = 2\pi r h \Delta r}$

$V =$[circumference][height][thickness]

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$. Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$ and let x_i^* be the midpoint of $[x_{i-1}, x_i]$, that is $x_i^* = (x_{i-1} + x_i)/2$. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i^*)$ is rotated about the y-axis, then the result is a cylindrical shell with average raduis x_i^*, height $f(x_i^*)$, and thikness $\Delta x_i = x_i - x_{i-1}$, so its volume is $V_i = 2\pi x_i^* f(x_i^*) \Delta x_i$.

The approximation to the volume V of S is
$V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.
Thus, the volume of S is

$$\boxed{V = \lim_{\|P\|\to 0} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i = 2\pi \int_a^b x f(x) dx}.$$

Example 3. Find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$, $y = 0$, $x = 0$, $x = 1$ about the y-axis.

The volume of the solid generated by rotating about the y-axis the region between the curves $y = f(x)$ and $y = g(x)$ from a to b $[f(x) \geq g(x)$ and $0 \leq a < b]$ is $\boxed{V = 2\pi \int_a^b x[f(x) - g(x)]dx}$.

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 4$, $x = 0$ about the y-axis.

The method of cylindrical shells also allows us to compute volumes of revolution about the x-axis. If we interchange the roles of x and y in the formula for the volume, then the volume of the solid generated by rotating the region bounded by $x = g(y)$, $x = 0$, $y = c$, and $y = d$ about the x-axis, is $\boxed{V = 2\pi \int_c^d y g(y) dy}$.

Example 5. Find the volume of the solid obtained by rotating the region bounded by $y^2 - 6y + x = 0$, $x = 0$ about the x-axis.

Example 6. Find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$, $y = 8x - 2x^2$ about $x = -2$.