Math 152, Fall 2008

Lecture 5.

09/09/2008

HW#2 is due Wednesday, September 10, 11:55 PM.

Chapter 7. **Applications of integration**
Section 7.4 **Work**

Mechanical work is the amount of energy transferred by a force.

If an object moves along a straight line with position function $s(t)$, then the force F on the object (in the same direction) is defined by Newton's Second Law of Motion

$$F = ma = m\frac{d^2s}{dt^2}$$

In case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves

$$W = Fd, \text{ work=force } \times \text{ distance}$$

Mechanical units in the U.S. customary and SI metric systems

Unit	U.S. customary system	SI metric system
distance	ft	m
mass	slug	kg
force	lb	N=$kg \cdot m/sec^2$
work	ft-lb	J=$N \cdot m$
g (Earth)	32 ft/sec^2	9.81 m/sec^2
the density of water	62.5 lb/ft^3	1000 kg/m^3

Example 1.

(a) Find the work done in pushing a car a distance of 8 m while exerting a constant force of 900 N.

(b) How much work is done by a weightlifter in raising a 60-kg barbell from the floor to the height of 2 m?

What happens if the force is variable?

Problem The object moves along the x-axis in the positive direction from $x = a$ to $x = b$ and at each point x between a and b a force $f(x)$ acts on the object, where f is continuous function. Find the work done in moving the object from a to b.

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$ and let $\Delta x_i = x_i - x_{i-1}$, and let x_i^* is in $[x_{i-1}, x_i]$. Then the force at x_i^* is $f(x_i^*)$. If $\|P\|$ is small, then Δx_i is small, and since f is continuous, the values of f do not change very much on $[x_{i-1}, x_i]$. In other words f is almost a constant on the interval and so work W_i that is done in moving the particle from x_{i-1} to x_i is $W_i \approx f(x_i^*)\Delta x_i$. We can approximate the total work by

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This approximation becomes better and better as $\|P\| \to 0$.

Therefore, we define the **work done in moving the object from a to b** as

$$W = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)dx$$

Example 2. When a particle is at a distance x meters from the origin, a force of $\cos(\pi x/3)$ N acts on it. How much work is done by moving the particle from $x = 1$ to $x = 2$.

Hooke's Law: The force required to maintain a spring stretched x units beyond its natural length is proportional to x $\boxed{f(x) = kx}$, where k is a positive constant (the **spring constant**).

Example 3. If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 in beyond its natural length?

Example 4. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all the water out over the side?

Section 7.5 **Average value of a function**

Let us try to compute the average value of a function $y = f(x)$, $a \leq x \leq b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b-a)/n$ and choose points x_i^* in successive subintervals. Then the average of the numbers $f(x_1^*)$, $f(x_2^*)$,...,$f(x_n^*)$, is

$$\frac{f(x_1^*) + f(x_2^*) + ... + f(x_n^*)}{n}$$

Since $n = (b-a)\Delta x$,

$$\frac{f(x_1^*) + f(x_2^*) + ... + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}(f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x)$$

The limiting value as $n \to \infty$ is

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a} \int_a^b f(x)dx$$

We define the **average value of** f on the interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 5. Find the average value of $f(x) = \sin^2 x \cos x$ on $[\pi/4, \pi/2]$.

Example 6. The temperature (in F^0) in a certain city t hours after 9 AM was approximated by the function

$$T(t) = 50 + 14\sin\frac{\pi t}{12}.$$

Find the average temperature during the perion from 9 AM to 9 PM.

Mean value theorem for integrals If f continuous on $[a, b]$, then there exist a number c in $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

The geometric interpretation of this theorem for *positive* functions $f(x)$, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as a region under the graph of f from a to b.

Example 7. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.