Math 152, Fall 2008

Lecture 9.

09/23/2008

The Week-In-Review session on Tuesday, Sept. 23, has been moved on Wednesday, Sept. 24 and starts at 7 PM in BLOC 158.

The Week-In-Review session covering the material of 4th week (trigonometric integrals, trigonometric substitution, partial fraction) starting at 7 PM.

After that (approximately starting from 8 PM) a 1st midterm review will be given.

HW#4 is due Wednesday, September 24, 11:55 PM.

In this section we show how to integrate any rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n$, $Q(x) = b_0 x^m + b_1 x^{m-1} + ... + b_m$ by expressing it as a sum of *partial fractions*, that we know how to integrate.

STEP 1. If f is improper ($m \geq n$), then we must divide Q into P by long divisions until a remainder $R(x)$ is obtained. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

STEP 2. Factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of *linear factors* of the form $ax + b$ and *irreducible quadratic factors* (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

STEP 3. Express the proper rational function $\dfrac{R(x)}{Q(x)}$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^j}$$

Four cases occur.

CASE I. $Q(x)$ **is a product of distinct linear factors.**

$$Q(x) = (a_1x + b_1)(a_2x + b_2)...(a_mx + b_m)$$

where no factor is repeated. Then there exist constants A_1, A_2,..., A_m such that

$$f(x) = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + ... + \frac{A_m}{a_mx + b_m}$$

Once the constants A_1, A_2,... , A_m are determined, the evaluation of $\dfrac{R(x)}{Q(x)}$ becomes a routine problem. The next examples will illustrate one method for finding these constants.

CASE II. $Q(x)$ **is a product of linear factors, some of which are repeated.**

Suppose the first linear factor $a_1x + b_1$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + ... + \frac{A_r}{(a_1x + b_1)^r}$$

Example 1. Evaluate $\displaystyle\int \frac{x^5 + 4}{x^3 - 5x^2 + 8x - 4}dx$

CASE III $Q(x)$ **contains irreducible quadratic factors none of which is repeated.**

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the corresponding fraction is

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

The term $\dfrac{Ax + B}{ax^2 + bx + c}$ can be integrating by completing the square in the denominator.

Example 2. Find $\displaystyle\int \frac{3x^3 - x^2 + 6x - 4}{(x^2 + 1)(x^2 + 2)}\,dx$

CASE IV $Q(x)$ **contains a repeated irreducible factor.**
If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then
instead of the single partial fraction $\dfrac{Ax + B}{ax^2 + bx + c}$, the sum

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each
of these terms can be integrated by completing the square and
making the tangent substitution.

Example 3. Evaluate $\displaystyle \int \frac{x^2 + x - 3}{(x^2 + 2x + 4)^2} dx$

Example 4. Write down the form of the partial fraction decomposition of the function (do not determine the numerical values of the coefficients).

(a) $\dfrac{2x+3}{(x^2-4x+3)(x^2-2x+1)^2}$

$$\frac{2x+3}{(x^2-4x+3)(x^2-2x+1)^2} = \frac{2x+3}{(x-3)(x-1)^5} =$$

$$\frac{A}{x-3} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{E}{(x-1)^4} + \frac{F}{(x-1)^5}$$

(b) $\dfrac{x+5}{(x^2-4)^2(x^2+2x+5)^2}$

$$\frac{x+5}{(x^2-4)^2(x^2+2x+5)^2} = \frac{x+5}{(x-2)^2(x+2)^2(x^2+2x+5)^2} =$$

$$\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{Ex+F}{x^2+2x+5} + \frac{Gx+H}{(x^2+2x+5)^2}$$

Strategy for integration

1. Simplify the intergand if possible

2. Look for an obvious substitution Try to find some function $u = g(x)$ un the integrand whose differential $du = g'(x)dx$ also occurs, apart from a constant factor.

3. Classify the integrand according to its form If steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.

(a) *Trigonometric functions.* If $f(x)$ is a product of powers of $\sin x$ and $\cos x$, of $\tan x$ and $\sec x$, of $\cot x$ and $\csc x$, then we use the substitutions recommended in section 8.2.

(b) *Rational functions.* If f is a rational function, we use the procedure of section 8.4 involving partial fractions.

(c) *Integration by parts.* if $f(x)$ is a product of a power of x (or a polynomial) and transcendental function (such as a trigonometric, exponential, logarithmic function), then we try integration by parts (section 8.1).

(d) *Radicals.* If $\sqrt{\pm x^2 \pm a^2}$ occurs, we use a trigonometric substitution according to the table in section 8.3.

4. Try again If the first three steps have not produced the answer, remember that there are basically two methods of integration: substitution and parts. Sometimes two or three methods are required to evaluate an integral.