Math 152, Fall 2008

Lecture 11.

09/30/2008

The due date for HW#5 has been moved on Saturday, October 4, 11:55 PM.

The correct answers for the multiple choice questions were:

Form A	1b, 2c, 3a, 4d, 5c, 6e, 7a, 8c, 9e, 10d
Form B	1a, 2d, 3a, 4e, 5a, 6a, 7c, 8d, 9e, 10e

Chapter 8. **Techniques of integration**
Section 8.8 **Approximate integration**

Sometimes it is impossible to find the exact value of the definite integral.
So, we need to find approximate values of definite integrals.
Recall, that the definite integral is defined as a limit of Riemann sums, so any Riemann sum could be used as approximation to the integral. In particular, let's take a partition of $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$, and let x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$ of partition. Then

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

If x_i^* is chosen to be the **left endpoint** of the ith subinterval, then $x_i^* = x_{i-1}$ and

$$\int_a^b f(x)dx \approx L_n = \frac{b-a}{n}\sum_{i=1}^{n} f(x_{i-1})$$

If x_i^* is chosen to be the **right endpoint** of the ith subinterval, then $x_i^* = x_i$ and

$$\int_a^b f(x)dx \approx R_n = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i)$$

If x_i^* is chosen to be the **midpoint** of the ith subinterval, then $x_i^* = \bar{x}_i = (x_{i-1} + x_i)/2$ and

$$\int_a^b f(x)dx \approx M_n = \frac{b-a}{n}\sum_{i=1}^{n} f(\bar{x}_i)$$

The approximations L_n, R_n, and M_n are called the **left endpoint approximation**, the **right endpoint approximation**, and the **midpoint approximation**, respectively.

The midpoint approximation M_n appears to be better than L_n or R_n.

Next method involves the **trapezoidal rule** which geometrically calculates the area of the trapezoid with base on the x-axis and heights $f(x_i)$ and $f(x_{i+1})$
The area of the trapezoid is

$$\frac{\Delta x}{2}(f(x_i) + f(x_{i+1}))$$

or the base times the average of the heights. Adding up all the trapezoids gives

$$\int_a^b f(x)dx \approx T_n = \frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+...+2f(x_{n-1})+f(x_n)],$$

here $x_i = a + i\Delta x$

Example 1. Use (a) the Midpoint Rule and (b) the Trapezoidal Rule to approximate the integral $\int_{\pi/2}^{\pi} \frac{\sin x}{x} dx$ with $n = 4$. (Round your answer to six decimal places.)

The **error** in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact.

Error bounds Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Example 2. How large do we have to choose n so that the approximations T_n and M_n to the integral $\int_0^1 e^x dx$ are accurate to within 0.00001?

Another rule for approximate integral results from using parabolas instead of straight line segments to approximate a curve.

We take a partition of $[a, b]$ into n subintervals of equal length $h = (b - a)/n$, but this time we assume that n is an **even** number.

Then on each consecutive pair of intervals we approximate the curve $y = f(x) > 0$ by a parabola that passes through three points $P_i(x_i, y_i)$, $P_{i+1}(x_{i+1}, y_{i+1})$, and $P_{i+2}(x_{i+2}, y_{i+2})$, here $y_i = f(x_i)$.

To simplify calculations, let's consider the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$.

We know that the equation of the parabola that passes trougth P_0, P_1 and P_2 is of form $p(x) = Ax^2 + Bx + C$, where A, B, and C are unknown constants.

To determine A, B, and C, we use that $p(x_0) = f(x_0) = y_0$, $p(x_1) = f(x_1) = y_1$, and $p(x_2) = f(x_2) = y_2$:

$$p(x_0) = Ax_0^2 + Bx_0 + C = y_0$$

$$p(x_1) = Ax_1^2 + Bx_1 + C = y_1$$

$$p(x_1) = Ax_2^2 + Bx_2 + C = y_2$$

Solving this system for A, B, and C gives

$$A = \frac{1}{2h^2}(y_0 - 2y_1 + y_2)$$

$$B = \frac{1}{2h}(y_2 - y_0)$$

$$C = y_1$$

The area under parabola from $x = -h$ to $x = h$ is

$$\int_{-h}^{h} (Ax^2 + Bx + C)dx = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Now, by shifting the parabola horizontally we do not change the area under it. This means that the area under the parabola trough P_0, P_1, and P_2 from $x = x_0$ to $x = x_2$ is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, the parabola trough P_i, P_{i+1}, and P_{i+2} from $x = x_i$ to $x = x_{i+2}$ is

$$\frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})$$

Adding all the areas gives

Simpson's Rule

$$\int_a^b f(x)dx \approx S_n = \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) +$$

$$4f(x_3) + 2f(x_4) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even.

Example 3. The speedometer reading (v) on a car was observed at 1-minute intervals and recorded in the following chart. Use the Simpson's Rule to estimate the distance travelled by car.

t(min)	0	1	2	3	4	5	6	7	8	9	10
v(mi/h)	40	42	45	49	52	54	56	57	57	55	56

Error bound for Simpson's Rule Suppose that $f^{(4)}(x) \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

Example 4. How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate within 0.00001?