Math 152, Fall 2008

Lecture 12.

10/02/2008

The due date for HW#5 is Saturday, October 4, 11:55 PM.

Chapter 8. **Techniques of integration**
Section 8.9 **Improper integrals**

In this section we extend the conception of a definite integral to the case where the interval is infinite and also to the case where integrand is unbounded.

Definition of an improper integral of type 1 (infinite intervals)

(a) If $\int\limits_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int\limits_a^\infty f(x)dx = \lim_{t \to \infty} \int\limits_a^t f(x)dx$$

provided this limit exists (as a finite number)

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_{t}^{b} f(x)dx$$

provided this limit exists (as a finite number)

The improper integrals in (a) and (b) are called **convergent** if the limit exist and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

where a is any real number

Example 1. For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

Example 2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x+3}}$ (b) $\displaystyle\int_{-\infty}^{-1} \frac{dx}{\sqrt[3]{x-1}}$ (c) $\int_{-\infty}^{\infty} (2x^2+x-1)dx$

(d) $\int_{-\infty}^{\infty} x^2 e^{-x^3} dx$ (e) $\displaystyle\int_0^\infty \frac{x}{(x+2)(x+3)}dx$

Definition of an improper integral of type 2 (discontinuous integrands)

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if this limit exists (as a finite number)

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if this limit exists (as a finite number)

The improper integrals in (a) and (b) are called **convergent** if the limit exist and **divergent** if the limit does not exist.

(c) If f has discontinuity at c $(a < c < b)$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example 3. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_0^3 \frac{1}{x\sqrt{x}}dx$ (b) $\int_{\pi/4}^{\pi/2} \sec^2 x dx$ (c) $\int_0^1 x \ln x dx$

Comparison theorem Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int\limits_a^\infty f(x)dx$ is convergent $\int\limits_a^\infty g(x)dx$ is convergent.

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Example 4. Use the Comparison Theorem to determine whether $\int\limits_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent or divergent.