Math 152, Fall 2008

Lecture 17.

10/21/2008

HW#8 is due Wednesday, October 22, 11:55 PM.

Chapter 10. **Infinite sequences and series**
Section 10.1 **Sequences**

A **sequence** is a list of numbers written in a definite order:

$$a_1, a_2, ..., a_n, ...$$

The number a_1 is called the *first term*, a_2 is the *second term*, a_n is the nth term. We will deal with infinite sequences and so each term a_n will have a successor a_{n+1}.

NOTATION: The sequence $a_1, a_2, ..., a_n, ...$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

Each sequence can be defined as a function whose domain is the set of positive integers ($a_n = f(n)$).

Some sequences can be defined by giving the formula for the nth term

$$\left\{n2^{-n}\right\}, \quad a_n = \frac{n+1}{n!}$$

another by writing out the terms of sequences

$$\left\{\frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}, \dots\right\}$$

n does not have to start at 1.

Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** or is **convergent**.
Otherwise, we say the sequence **diverges** or is **divergent**

Limit Laws If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} [a_n - b_n] = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{x\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} c = c$

The Squeeze Theorem If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 1. Find the limit

(a) $\lim_{n\to\infty} \frac{n-2}{3n+1}$ (b) $\lim_{n\to\infty} \frac{\ln(n^2)}{n}$

(c) $\lim_{n\to\infty} (\sqrt{n+2} - \sqrt{n})$

(d) $\lim_{n\to\infty} \frac{(-1)^n}{n!}$ (e) $\lim_{n\to\infty} \frac{n!}{n^n}$

Example 2. For what values of r is the sequence $\{r^n\}$ convergent?

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Example 3. Determine whether the sequence is increasing, decreasing, or not monotonic.

(a) $a_n = \dfrac{1}{3n+5}$ (b) $a_n = 3 + \dfrac{(-1)^n}{n}$ (c) $a_n = \dfrac{n-2}{n+2}$

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \qquad \text{for all } n \geq 1$$

It is **bounded below** if there is a number m such that

$$a_n \geq m \qquad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**

Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

Example 4. Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - 1/a_n$$

is increasing and $a_n < 3$ for all n.