Math 152 **Fall 2008** **Review Before Test 1**

The Test 1 will be held on Thursday, Sept. 25, at 7:30-9:30 PM in HELD 105.

It will cover sections 6.4, 6.5, 7.1 – 7.5, 8.1 – 8.4.

Calculators are not allowed on the test.

Chapter 7. **Applications of integration**

Section 7.1 **Areas between curves**

The area of the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, is $\boxed{A = \int_a^b |f(x) - g(x)|dx}$

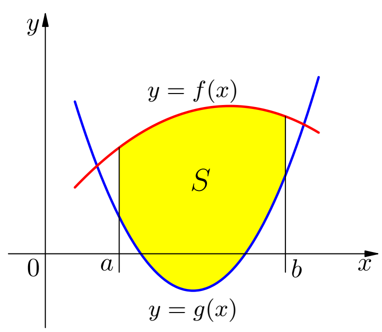

Fig.1 $\quad S = \{(x, y) : a \le x \le b, g(x) \le y \le f(x)\}$

If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous functions and $f(y) \ge g(y)$ for all y in $[c, d]$, then its area is $\boxed{A = \int_c^d [f(y) - g(y)]dy}$

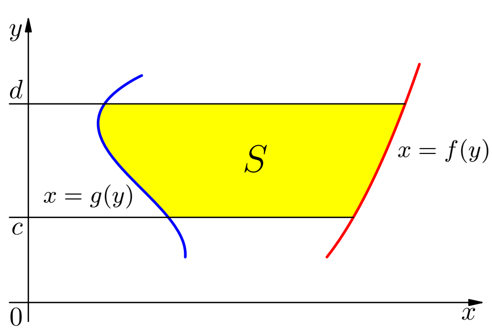

Fig.2 $\quad S = \{(x, y) : g(y) \le x \le f(y), c \le y \le d\}$

Example 1. Find the area of the region bounded by $y = \sin x$, $y = -\cos x$, $x = 0$, $x = \pi$.

Section 7.2 **Volume**
Section 7.3 **Volumes by cylindrical shells**

Definition of volume Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is an integrable function, then the **volume** of S is $\boxed{V = \int_a^b A(x)dx}$

Example 2. Let S is a solid with the base $\{(x, y) : x^2 \leq y \leq 1\}$, and whose cross-sections perpendicular to the y-axis are equilateral triangles. Compute the volume of S.

Let S be the solid obtained by revolving the plane region $\mathcal{R}$ bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the x-axis, then its volume $\boxed{V = \pi \int_a^b [f(x)]^2 dx}$ (*disk method*)

The volume of the solid generated by rotating the region bounded by $x = g(y)$, $x = 0$, $y = c$, and $y = d$ about the x-axis, is $\boxed{V = 2\pi \int_c^d y g(y) dy}$. (*cylindrical shells method*)

The region bounded by the curves $x = g(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the y-axis, then the corresponding volume of revolution is $\boxed{V = \pi \int_c^d [g(y)]^2 dy}$ (*disk method*)

Let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$. Then its volume is $\boxed{V = 2\pi \int_a^b x f(x) dx}$ (*cylindrical shells method*)

Let S be the solid generated when the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ (where $f(x) \geq g(x)$ for all x in $[a, b]$) is rotated about the x-axis. Then the volume of S is $\boxed{V = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\} dx}$

The volume of the solid generated by rotating about the y-axis the region between the curves $y = f(x)$ and $y = g(x)$ from a to b $[f(x) \geq g(x)$ and $0 \leq a < b]$ is $\boxed{V = 2\pi \int_a^b x[f(x) - g(x)] dx}$.

Example 3. (a) Let $\mathcal{R}_a$ be the region bounded by $y = x^2$, $y = 4$, $x = 0$, $x = 2$. Use the method of disks to find the volume of the solid generated by rotating $\mathcal{R}_a$ about y-axis.

(b) Let $\mathcal{R}_b$ be the region bounded by $y = x^2$, $y = 0$, $x = 1$, $x = 2$. Use the method of cylindrical shells to find the volume of the solid generated by rotating $\mathcal{R}_b$ about the line $x = 4$.

Section 7.4 **Work**

The **work done in moving the object from a to b** is $\boxed{W = \int\limits_a^b f(x)dx}$

Example 4. Suppose that 2 J of work are needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch it from 35 cm to 40 cm?

Example 5. A tank full of water has a shape of paraboloid of revolution (its shape is obtained by rotating a parabola about a vertical axis). If its height is 4 m and the radius of top is 4 m, find the work required to pump the water out of tank.

Section 7.5 **Average value of a function**

The **average value of f** on the interval $[a, b]$ is $\boxed{f_{ave} = \dfrac{1}{b-a}\int\limits_a^b f(x)dx}$

Example 6. Find the average value of the function $f(x) = x^3$ on the interval $[2, 4]$.

Example 7. Suppose that f is continuous function defined on $[0, \infty)$, and the average value of f over the interval $[0, t]$ is $t - 3$ for every $t > 0$. Find f.

Chapter 8. **Techniques of integration**

Strategy for integration

1. Simplify the intergand if possible

2. Look for an obvious substitution Try to find some function $u = g(x)$ un the integrand whose differential $du = g'(x)dx$ also occurs, apart from a constant factor.

$$\int f(g(x))g'(x)dx = \int f(u)du$$

$$\int\limits_a^b f(g(x))g'(x)dx = \int\limits_{g(a)}^{g(b)} f(u)du$$

Example 8. Find $\displaystyle\int\limits_0^1 \frac{x^2}{(2x+1)^{10}}dx$.

3. Classify the integrand according to its form If steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.

(a) *Trigonometric functions.*

How to evaluate $\int \sin^m x \cos^n x \; dx$

(a) if the power of cosine is odd, save one cosine factor and use $\boxed{\cos^2 x = 1 - \sin^2 x}$ to express the remaining factors in terms of sine. Then substitute $u = \sin x$

(b) if the power of sine is odd, save one sine factor and use $\boxed{\sin^2 x = 1 - \cos^2 x}$ to express the remaining factors in terms of cosine. Then substitute $u = \cos x$

(c) if both m and n are even, use the half-angle identities $\boxed{\sin^2 x = \frac{1}{2}(1 - \cos 2x)}$, $\boxed{\cos^2 x = \frac{1}{2}(1 + \cos 2x)}$, $\boxed{\sin x \cos x = \frac{1}{2} \sin 2x}$.

How to evaluate $\int \tan^m x \sec^n x \; dx$

(a) if the power of secant is even, save a factor of $\sec^2 x$ and use $\boxed{\sec^2 x = 1 + \tan^2 x}$ to express the remaining factors in terms of $\tan x$. Then substitute $u = \tan x$.

(b) if the power of tangent is odd, save a factor of $\tan x \sec x$ and use $\boxed{\tan^2 x = \sec^2 x - 1}$ to express the remaining factors in terms of $\sec x$. Then substitute $u = \sec x$.

To evaluate the integrals $\int \sin mx \; \cos nx \; dx$, $\int \sin mx \; \sin nx \; dx$, $\int \cos mx \; \cos nx \; dx$, use the identities:

$\boxed{\sin\alpha \; \cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]}$,

$\boxed{\sin\alpha \; \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]}$,

$\boxed{\cos\alpha \; \cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]}$

Example 9. Evaluate

(a) $\int \sin^4 x \cos^2 x \; dx$

SOLUTION. $\displaystyle \int \sin^4 x \cos^2 x \; dx = \frac{1}{4} \int \sin^2 x \sin^2 2x \; dx = \frac{1}{4} \int \frac{1 - \cos 2x}{2} \sin^2 2x \; dx =$

$\displaystyle \frac{1}{8} \int (\sin^2 2x - \cos 2x \sin^2 2x) \; dx = \frac{1}{8} \int \sin^2 2x \; dx - \frac{1}{8} \int \cos 2x \sin^2 2x \; dx =$

Let's make a substitution in the second integral $\sin 2x = u$, then $du = 2\cos 2x dx$

$\displaystyle \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx - \frac{1}{16} \int u^2 du = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x \right) - \frac{1}{16} \frac{u^3}{3} + C =$

$\displaystyle \frac{1}{16} x - \frac{1}{64} \sin 4x - \frac{1}{48} \sin^3 2x + C$

(b) $\displaystyle \int_0^{\pi/2} \sin^2 x \cos^3 x \; dx$

SOLUTION. $\displaystyle \int_0^{\pi/2} \sin^2 x \cos^3 x \; dx = \int_0^{\pi/2} \sin^2 x \cos^2 x \cos x \; dx = \int_0^{\pi/2} \sin^2 x (1 - \sin^2 x) \cos x \; dx =$

$$\left| \begin{array}{ll} u=\sin x & x=0\to u=0 \\ du=\cos x dx & x=\pi/2\to u=1 \end{array} \right| = \int_0^1 u^2(1-u^2)du = \int_0^1 (u^2-u^4)du = \left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Big|_0^1 = \frac{2}{15}$$

(c) $\int \tan^3 x \sec^3 x \; dx$

SOLUTION. $\int \tan^3 x \sec^3 x \; dx = \int \tan^2 x \sec^2 x (\tan x \sec x) \; dx = \int (\sec^2 x+1)\sec^2 x(\tan x \sec x) \; dx =$

$$\left| \begin{array}{c} u=\sec x \\ du=\sec x\tan x \; dx \end{array} \right| = \int (u^2+1)u^2 du = \int (u^4+u^2)du = \frac{u^5}{5}+\frac{u^3}{3}+C = \frac{\sec^5 x}{5}+\frac{\sec^3 x}{3}+C$$

(d) $\int \cos 2x \sin x \; dx$

SOLUTION. $\int \cos 2x \sin x \; dx = \frac{1}{2}\int (\sin x+\sin 3x)dx = \frac{1}{2}(-\cos x - \frac{1}{3}\cos 3x)+C$

(b) *Rational functions.* If f is a rational function, then $f(x) = \frac{P(x)}{Q(x)}$, where $P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, $Q(x) = b_0x^m + b_1x^{m-1} + ... + b_m$ by expressing it as a sum of *partial fractions*, that we know how to integrate.

STEP 1. If f is improper ($m \geq n$), then we must divide Q into P by long divisions until a remainder $R(x)$ is obtained. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

STEP 2. Factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of *linear factors* of the form $ax+b$ and *irreducible quadratic factors* (of the form ax^2+bx+c, where $b^2-4ac<0$).

STEP 3. Express the proper rational function $\frac{R(x)}{Q(x)}$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

Four cases occur.

CASE I. $Q(x)$ **is a product of distinct linear factors.**

$$Q(x) = (a_1x+b_1)(a_2x+b_2)...(a_mx+b_m)$$

where no factor is repeated. Then there exist constants A_1, A_2,..., A_m such that

$$f(x) = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ... + \frac{A_m}{a_mx+b_m}$$

CASE II. $Q(x)$ **is a product of linear factors, some of which are repeated.**

Suppose the first linear factor $a_1x + b_1$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + ... + \frac{A_r}{(a_1x + b_1)^r}$$

CASE III $Q(x)$ **contains irreducible quadratic factors none of which is repeated.**

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the corresponding fraction is

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

The term $\dfrac{Ax + B}{ax^2 + bx + c}$ can be integrating by completing the square in the denominator.

CASE IV $Q(x)$ **contains a repeated irreducible factor.**

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction $\dfrac{Ax + B}{ax^2 + bx + c}$, the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of these terms can be integrated by completing the square and making the tangent substitution.

Example 10. Find $\displaystyle\int \frac{x^3 + 1}{x^3 - x^2}dx$

SOLUTION. $\dfrac{x^3 + 1}{x^3 - x^2} = 1 + \dfrac{1 + x^2}{x^3 - x^2}$

$$\frac{1 + x^2}{x^3 - x^2} = \frac{1 + x^2}{x^2(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}$$

$$\left.\frac{1 + x^2}{x^2(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}\right| \times x^2(x - 1)$$

$1 + x^2 = Ax(x - 1) + B(x - 1) + Cx^2$

$x = 0 : \quad 1 = B$

$x = 1 : \quad 2 = C$

$x = -1 : \quad 2 = (-2)(-1)A - 2B + C$

Thus, $A = 1$, $B = 1$, $C = 2$, and

$$\frac{1 + x^2}{x^2(x - 1)} = \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x - 1}$$

Then

$$\int \frac{x^3+1}{x^3-x^2}dx = \int \left(1+\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x-1}\right)dx = x+\ln|x|-\frac{1}{x}+2\ln|x-1|+C$$

(c) *Integration by parts.* If $f(x)$ is a product of a power of x (or a polynomial) and transcendental function (such as a trigonometric, exponential, logarithmic function), then we try integration by parts.

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

$$\int_a^b f'(x)g(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx$$

Example 11. Evaluate $\int_0^2 e^{\sqrt{x}}dx$

SOLUTION. $\int_0^2 e^{\sqrt{x}}dx = \left| \begin{matrix} u=\sqrt{x}\to x=u^2 & x=0\to u=0 \\ dx=2udu & x=2\to u=\sqrt{2} \end{matrix} \right| = 2\int_0^{\sqrt{2}} ue^u du =$

$\left| \begin{matrix} f'(u)=e^u & f(u)=e^u \\ g(u)=u & g'(u)=1 \end{matrix} \right| = 2(ue^u|_0^{\sqrt{2}} - \int_0^{\sqrt{2}} e^u du) = 2(\sqrt{2}e^{\sqrt{2}} - e^u|_0^{\sqrt{2}}) = 2(\sqrt{2}e^{\sqrt{2}} - e^{\sqrt{2}}+1)$

(d) *Radicals.* If $\sqrt{\pm x^2 \pm a^2}$ occurs, we use a trigonometric substitution according to the following table

Table of trigonometric substitutions

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x=a\sin t, -\pi/2\le t\le \pi/2$	$1-\sin^2 t=\cos^2 t$
$\sqrt{a^2+x^2}$	$x=a\tan t, -\pi/2< t< \pi/2$	$1+\tan^2 t=\sec^2 t$
$\sqrt{x^2-a^2}$	$x=a\sec t, 0\le t\le \pi/2$ or $\pi\le t\le 3\pi/2$	$\sec^2 t-1=\tan^2 t$

Example 12. Find $\int \sqrt{1+4x-x^2}dx$.

SOLUTION. Let's complete the square under the root sign:

$1+4x-x^2 = -(x^2-4x-1) = -(x^2-4x+4-4-1) = -((x-2)^2-5) = 5-(x-2)^2$

$\int \sqrt{1+4x-x^2}dx = \int \sqrt{5-(x-2)^2}dx = \left| \begin{matrix} x-2=u \\ dx=du \end{matrix} \right| = \int \sqrt{5-u^2}du = \left| \begin{matrix} u=\sqrt{5}\sin t \\ du=\sqrt{5}\cos t dt \\ \sqrt{5-u^2}=\sqrt{5}\cos t \end{matrix} \right| =$

$\int \sqrt{5}\cos t\sqrt{5}\cos t dt = 5\int \cos^2 t dt = 5\int \frac{1+\cos 2t}{2}dt = \frac{5}{2}\left(t+\frac{1}{2}\sin 2t\right)+C$

Since $\sin t = \frac{u}{\sqrt{5}}$, then $\cos t = \sqrt{1-\sin^2 t} = \frac{1}{\sqrt{5}}\sqrt{5-u^2}$ and

$\sin 2t = 2\sin t\cos t = 2\frac{u}{\sqrt{5}}\frac{1}{\sqrt{5}}\sqrt{5-u^2} = \frac{2u}{5}\sqrt{5-u^2}$

Thus,

$$\frac{5}{2}\left(t+\frac{1}{2}\sin 2t\right)+C=\frac{5}{2}\left(\arcsin\frac{u}{\sqrt{5}}+\frac{2}{5}u\sqrt{5-u^2}\right)+C=$$

$$\frac{5}{2}\left(\arcsin\frac{x-2}{\sqrt{5}}+\frac{2}{5}(x-2)\sqrt{5-(x-2)^2}\right)+C$$

4. Try again If the first three steps have not produced the answer, remember that there are basically two methods of integration: substitution and parts. Sometimes two or three methods are required to evaluate an integral.