Math 152 **Fall 2008** **Review Before Test 2**

The Test 2 will be held on Thursday, Oct. 23, at 7:30-9:30 PM in HELD 105.

It will cover sections 8.8, 8.9, 9.1-9.6. Calculators are not allowed on the test.

Chapter 8. **Techniques of integration**
Section 8.8 **Approximate integration**

Trapezoidal Rule

$$\int_a^b f(x)dx \approx T_n = \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)],$$

here $x_i = a + i\Delta x$

Simpson's Rule

$$\int_a^b f(x)dx \approx S_n = \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) +$$

$$4f(x_3) + 2f(x_4) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even.

Example 1. Use (a) the Trapezoidal Rule and (b) the Simpson's Rule to approximate the integral

$$\int_1^3 x^2 dx$$

with $n = 4$.

Section 8.9 **Improper integrals**

Definition of an improper integral of type 1 (infinite intervals)

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number)

(b) If $\int_t^b f(x)dx$ exists for every number $t \le b$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_{t}^{b} f(x)dx$$

provided this limit exists (as a finite number)

The improper integrals in (a) and (b) are called **convergent** if the limit exist and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

where a is any real number.

$$\boxed{\int_1^\infty \frac{1}{x^p} = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p \le 1 \end{cases}}$$

Definition of an improper integral of type 2 (discontinuous integrands)

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_{a}^{b} f(x)dx = \lim_{t\to b^-} \int_{a}^{t} f(x)dx$$

if this limit exists (as a finite number)

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_{a}^{b} f(x)dx = \lim_{t\to a^+} \int_{t}^{b} f(x)dx$$

if this limit exists (as a finite number)

The improper integrals in (a) and (b) are called **convergent** if the limit exist and **divergent** if the limit does not exist.

(c) If f has discontinuity at c $(a < c < b)$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$$

Example 2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_1^\infty \frac{\sqrt{x}}{2x+3}dx$ (b) $\int_1^\infty \frac{e^{-x}}{x}dx$ (c) $\int_0^4 \frac{1}{x^2+x-6}dx$

Chapter 9. **Further applications of integration**
Section 9.1 **Differential equations**

Definition. Equation that contains some derivatives of an unknown function is called a **differential equation**.

Definition. The **order** of a differential equation is the order of the highest-order derivatives present in equation.

Definition. A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

When we are asked to solve a differential equation we are expected to find all possible solutions to this equation.

We are interested in first order differential equations

$$\frac{dy}{dx} = f(x, y)$$

In many problems we need to find the particular solution to the equation

$$\frac{dy}{dx} = f(x, y)$$

that satisfies a condition

$$y(x_0) = y_0$$

The condition $y(x_0) = y_0$ is called the **initial condition** . The problem of finding a solution to the differential equation that satisfies the initial condition is called an **initial value problem**.

A separable equation is a first order differential equation that can be written in the form

$$\frac{dy}{dx} = f(x)g(y) \quad \text{or} \quad \frac{dy}{dx} = \frac{f(x)}{h(y)}$$

To solve this equation we rewrite it in the differential form

$$h(y)dy = f(x)dx$$

Then we integrate both sides of the equation

$$\int h(y)dy = \int f(x)dx$$

The **general solution** to this equation is

$$H(y) = F(x) + C$$

where $H(y)$ is an antiderivative of $h(y)$, $F(x)$ is an antiderivative of $f(x)$, C is a constant.

Section 9.2 **First order linear equations**

A first order linear differential equation is an equation that can be written in the form

$$y' + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions.

To solve the first order linear equation
(a) Find the integrating factor $I(x) = e^{\int P(x)dx}$
(b) Integrate the equation

$$\frac{d}{dx}[I(x)y] = I(x)Q(x)$$

and solve it for y by dividing by $I(x)$.

Example 3. Solve the differential equation/initial value problem.

(a) $y' = \dfrac{\ln x}{xy + xy^3}$

(b) $xy' - 3y = x^2$, $x > 0$, $y(1) = 0$

Chapter 9. **Further applications of integration**
Section 9.3 **Arc length**

If a curve C is defined by the equations

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

then its length is

$$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

If the curve C is given by the equation

$$y = y(x), \quad a \le x \le b, \quad \text{then} \quad L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

If the curve C is given by the equation

$$x = x(y), \quad c \le y \le d, \quad \text{then} \quad L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$$

Example 4. Find the length of the curve $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$

Section 9.4 **Area of a surface of revolution**

A surface of revolution is formed when a curve is rotated about a line.

Consider the surface which is obtained by rotating the curve $y = f(x)$, $a \le x \le b$ about the x-axis, $f(x) > 0$ for all x in $[a, b]$ and $f'(x)$ is continuous.

The area of the surface is

$$S_X = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx = 2\pi \int_a^b f(x)\sqrt{1 + \left[\frac{dy}{dx}\right]^2}dx$$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for the surface area is

$$S_X = 2\pi \int_c^d y\sqrt{1 + [g'(y)]^2}dy = 2\pi \int_c^d y\sqrt{1 + \left[\frac{dx}{dy}\right]^2}dy$$

If a curve C is defined by the equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, then the area of the surface generated by rotating C about x-axis is

$$S_X = 2\pi \int_a^b y(t)\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}dt$$

Example 5. Find the area of the surface obtained by rotating the curve $y = \sin x$, $0 \le x \le \pi$ about x-axis

If the curve is given as $y = f(x)$, $a \le x \le b$, the area of the surface generated by rotating the curve about y-axis is

$$S_Y = 2\pi \int_a^b x\sqrt{1 + \left[\frac{dy}{dx}\right]^2}dx$$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the area of the surface generated by rotating the curve about y-axis is

$$S_Y = 2\pi \int_c^d g(y)\sqrt{1 + \left[\frac{dx}{dy}\right]^2}dy$$

If the curve is defined by the equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, then the area of the surface generated by rotating the curve about y-axis is

$$S_Y = 2\pi \int_a^b x(t)\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}dt$$

Example 6. Find the area of the surface obtained by rotating the curve $x = y^3$, $1 \le y \le 2$ about y-axis

Section 9.5 **Moments and centers of mass**

Consider a flat plate with uniform density ρ that occupies a region $R = \{(x, y) : a \leq x \leq b, g(x) \leq y \leq f(x)\}$.

The moment of R about x-axis is

$$M_x = \frac{1}{2}\rho \int_a^b \{[f(x)]^2 - [g(x)]^2\}dx$$

The moment of R about y-axis is

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx$$

The centroid of R is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\int_a^b x[f(x) - g(x)]dx}{\int_a^b [f(x) - g(x)]dx}$$

$$\bar{y} = \frac{\int_a^b \{[f(x)]^2 - [g(x)]^2\}dx}{\int_a^b [f(x) - g(x)]dx}$$

Example 7. Find the centroid of the region bounded by $y = \sqrt{x}$ and $y = x$.

Section 9.6 **Hydrostatic pressure and force**

Example 8. A tank contains water. The end of the tank is vertical and has the shape of a isosceles triangle with the base 4 m and the heigh 6 m. Find the hydrostatic force against the end of the tank.