1. Find the area of the region bounded by $y = x^2 + 1$, $y = 3 - x^2$, $x = -2$, and $x = 2$.

2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 1$, $y = 0$, $x = 1$, $x = 2$ about the x-axis.

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 0$. $x = 1$, $x = 2$ about

   (a) the y-axis

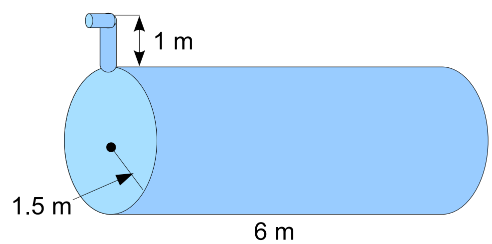
   (b) $x = 4$

4. The base of solid S is the triangular region with vertices (0,0), (2,0), and (0,1). Cross-sections perpendicular to the x-axis are semicircles. Find the volume of S.

5. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft hight. How much work is done in pulling the rope to the top of the building?

6. A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

7. A tank is full of water. Find the work required to pump the water out the outlet.



8. Find the average value of $f = \sin^2 x \cos x$ on $[-\pi/2, \pi/4]$.

9. Evaluate the integral

   (a) $\displaystyle\int t^2 \cos(1 - t^3)dt$

   (b) $\displaystyle\int \frac{x^2}{\sqrt{1-x}}dx$

   (c) $\displaystyle\int_0^1 x^2 e^{-x} dx$

   (d) $\displaystyle\int \sin^3 x \cos^4 x \; dx$

   (e) $\displaystyle\int_0^{\pi/8} \sin^2(2x)\cos^3(2x) \; dx$

   (f) $\displaystyle\int \sin^2 x \cos^4 x \; dx$

(g) $\displaystyle\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

(h) $\displaystyle\int \tan x \sec^3 x \, dx$

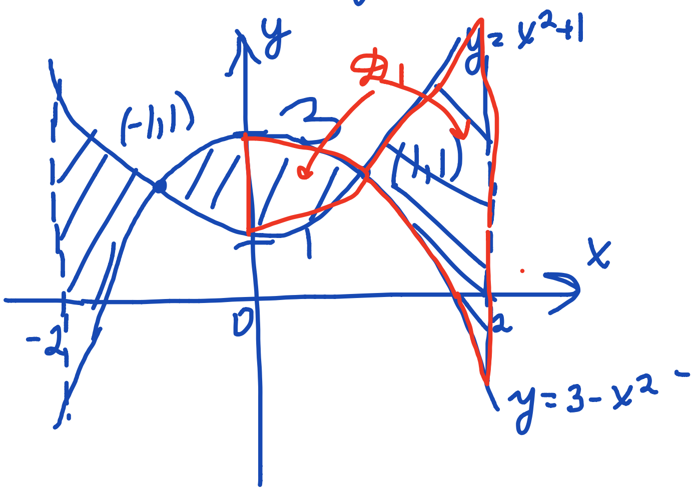
(i) $\displaystyle\int \sin 3x \cos x \, dx$

(j) $\displaystyle\int_0^{2/3} x^3\sqrt{4-9x^2} \, dx$

(k) $\displaystyle\int \frac{1}{\sqrt{9x^2+6x-8}} dx$

(l) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}} dx$

#1. $y = x^2+1,\ y = 3-x^2,\ x=-2,\ x=2$



Points of intersection:

$x^2+1 = 3-x^2$

$2x^2 = 2, \quad x^2 = 1 \Rightarrow x = \pm 1$

$(-1,1)$ and $(1,1)$ $\quad y = 1$

$A = 2\left(\int_0^1 [(3-x^2)-(x^2+1)]\,dx + \int_1^2 ((1+x^2)-(3-x^2))\,dx\right)$

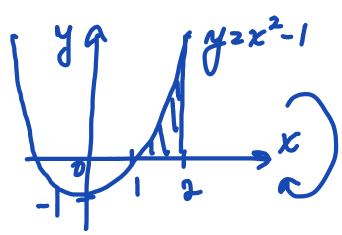
$= 2\left[\int_0^1 (2-2x^2)\,dx + \int_1^2 (2x^2-2)\,dx\right]$

$= 2\left[\left(2x - \frac{2x^3}{3}\right)\Big|_0^1 + \left(\frac{2x^3}{3} - 2x\right)\Big|_1^2\right]$

$= 2\left[2 - \frac{2}{3} + \frac{2\cdot 8}{3} - 4 - \frac{2}{3} - 2\right]$

$= \boxed{8}$

#2. $y = x^2-1,\ y=0,\ x=1,\ x=2$ about the x-axis



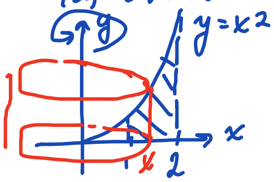
disks: $V = \pi\int_1^2 [x^2-1]^2\,dx$

$= \pi\int_1^2 (x^4 - 2x^2 + 1)\,dx$

$= \pi\left(\frac{x^5}{5} - 2\frac{x^3}{3} + x\right)\Big|_1^2 = \boxed{\frac{38\pi}{15}}$

#3. $y = x^2$, $y = 0$, $x = 1$, $x = 2$

(a) about the y-axis



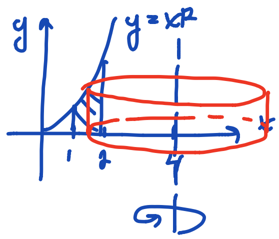
cylindrical shells:

$$V = 2\pi \int_1^2 r\,h\,dx$$

$r = x$, $h = x^2$

$$V = 2\pi \int_1^2 x\,x^2\,dx = 2\pi \int_1^2 x^3\,dx = 2\pi \left.\frac{x^4}{4}\right]_1^2 = \boxed{\frac{15\pi}{2}}$$

(b) about $x = 4$



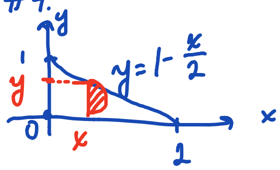
cylindrical shells:

$r = 4 - x$, $h = x^2$

$$V = 2\pi \int_1^2 (4 - x)x^2\,dx = 2\pi \int_1^2 (4x^2 - x^3)\,dx$$

$$= 2\pi\left(\frac{4x^3}{3} - \frac{x^4}{4}\right)\Big|_1^2 = \boxed{\frac{67\pi}{6}}$$

#4.



$0 \le x \le 2$

$$V = \int_0^2 A(s)\,dx$$

$A(s) = \frac{1}{2}\pi r^2$

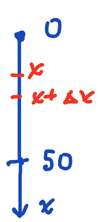
$r = \frac{1}{2}y$, $y = 1 - \frac{x}{2}$

$$A(s) = \frac{1}{2}\pi\,\frac{1}{4}\left(1 - \frac{x}{2}\right)^2 = \frac{\pi}{8}\left(1 - \frac{x}{2}\right)^2$$

$$V = \frac{\pi}{8}\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \frac{\pi}{8}\int_0^2 \left(1 - x + \frac{x^2}{4}\right) dx = \frac{\pi}{8}\left(x - \frac{x^2}{2} + \frac{x^3}{12}\right)\Big|_0^2$$

$$= \frac{\pi}{8}\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \boxed{\frac{\pi}{12}}$$

#5.



$0 \le x \le 50$

The portion of the rope from x to $x + \Delta x$ weighs $0.5\Delta x$ (lb) and must be lifted x (ft)

$$W = \int_0^{50} \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^{50} = \boxed{625 \text{ (ft-lb)}}$$

#6. $F = kx$

$20\text{ cm} \to 0$

$30\text{ cm} \to 30-20 = 10\,(\text{cm}) = 0.1\,(\text{m})$

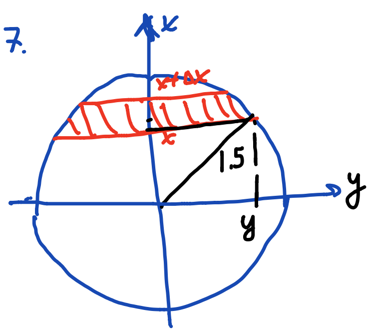
$25 = k(0.1) \Rightarrow k = 250$

$F = 250x$

$25\text{ cm} \to 25-20 = 5\,(\text{cm}) = 0.05\,(\text{m})$

$$W = \int_0^{0.05} 250x\,dx = \frac{250}{2}x^2\Big]_0^{0.05} = \boxed{3.125\,(J)}$$

#7.



$-1.5 \le x \le 1.5$

a slice of water between x and $x+\Delta x$ weighs $(10^3)(9.81)(6)\,y\,\Delta x$

$y^2 = 1.5^2 - x^2$

$y = \sqrt{2.25 - x^2}$

weight $= (10^3)(9.81)(6)\sqrt{2.25-x^2}\,\Delta x$

The slice must be lifted by $1.5 - x + 1 = 2.5 - x$

$$W = \int_{-1.5}^{1.5} (10^3)(9.8)(6)\sqrt{2.25-x^2}\,(2.5-x)\,dx$$

$$= (10^3)(9.8)(6)(2.5)\int_{-1.5}^{1.5}\sqrt{2.25-x^2}\,dx - 10^3(9.8)(6)\underbrace{\int_{-1.5}^{1.5} x\sqrt{2.25-x^2}\,dx}_{=0}$$

$x = 1.5\sin t$, $t = \sin^{-1}\left(\frac{x}{1.5}\right)$

$dx = 1.5\cos t\,dt$

$\sqrt{2.25-x^2} = \sqrt{2.25 - 2.25\sin^2 t}$

$= 1.5\cos t$

$-1.5 \to \sin^{-1}\left(\frac{-1.5}{1.5}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$

$1.5 \to \sin^{-1}\left(\frac{1.5}{1.5}\right) = \sin^{-1}(1) = \frac{\pi}{2}$

$u = 2.25 - x^2$

$du = -2x\,dx$

$-1.5 \to \sqrt{2.25-(-1.5)^2} = 0$

$1.5 \to \sqrt{2.25-(1.5)^2} = 0$

$$= 10^3(9.8)(6)(2.5)\int_{-\pi/2}^{\pi/2} 1.5\cos t\,(1.5)\cos t\,dt = 10^3(9.8)(6)(2.5)(2.25)\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt$$

$$=(10^3)(9.8)(6)(2.5)(2.25)\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2t}{2}\,dt = 165375\left(t+\frac{1}{2}\sin 2t\right)\Big|_{-\pi/2}^{\pi/2}$$

$$=165375\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{1}{2}\sin\pi-\frac{1}{2}\sin(-\pi)\right)$$

$$=\boxed{165375\,\pi}$$

#8. $f=\sin^2 x\cos x$ on $\left[-\frac{\pi}{2},\frac{\pi}{4}\right]$

$$f_{ave}=\frac{1}{\frac{\pi}{4}-\left(-\frac{\pi}{2}\right)}\int_{-\pi/2}^{\pi/4}\sin^2 x\cos x\,dx=\left|\begin{array}{l} u=\sin x\\ du=\cos x\,dx\\ -\frac{\pi}{2}\to\sin\left(-\frac{\pi}{2}\right)=-1\\ \frac{\pi}{4}\to\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\end{array}\right|$$

$$=\frac{4}{3\pi}\int_{-1}^{\frac{\sqrt{2}}{2}}u^2\,du=\frac{4}{3\pi}\,\frac{u^3}{3}\Big]_{-1}^{\frac{\sqrt{2}}{2}}=\boxed{\frac{4}{9\pi}\left(\frac{2\sqrt{2}}{8}+1\right)}$$

#9.

(a) $$\int t^2\cos(1-t^3)\,dt\ \left|\begin{array}{l}u=1-t^3\\ du=-3t^2dt\end{array}\right|=-\frac{1}{3}\int\cos u\,du=-\frac{1}{3}\sin u+C$$

$$=\boxed{-\frac{1}{3}\sin(1-t^3)+C}$$

(b) $$\int\frac{x^2}{\sqrt{1-x}}\,dx=\left|\begin{array}{l}1-x=u,\quad x=1-u\\ du=-dx\\ x^2=(1-u)^2\end{array}\right|=\int\frac{(1-u)^2}{\sqrt{u}}(-du)$$

$$=-\int\frac{1-2u+u^2}{\sqrt{u}}\,du=-\int\left(u^{-1/2}-2u^{1/2}+u^{3/2}\right)du$$

$$=-\left[\frac{u^{1/2}}{1/2}-2\,\frac{u^{3/2}}{3/2}+\frac{u^{5/2}}{5/2}\right]+C=-2u^{-1/2}+\frac{4}{3}u^{3/2}-\frac{2}{5}u^{5/2}+C$$

$$=\boxed{-2(1-x)^{-1/2}+\frac{4}{3}(1-x)^{3/2}-\frac{2}{5}(1-x)^{5/2}+C}$$

(c) $\int_0^1 x^2 e^{-x}\,dx \quad \left| \begin{array}{ll} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{array} \right| = -x^2 e^{-x}\Big|_0^1 - \int_0^1 (2x)(-e^{-x})\,dx$

$= -e^{-1} + 2\int_0^1 x e^{-x}\,dx = \left| \begin{array}{ll} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{array} \right|$

$= -e^{-1} + 2\left[x(-e^{-x})\Big|_0^1 - \int_0^1 (-e^{-x})\,dx\right]$

$= -e^{-1} + 2\left[-e^{-1} + \int_0^1 e^{-x}\,dx\right] = -3e^{-1} - 2e^{-x}\Big|_0^1 = -3e^{-1} - 2(e^{-1} - 1)$

$= \boxed{2 - 5e^{-1}}$

(d) $\int \sin^3 x \cos^4 x\,dx = \int \sin x \sin^2 x \cos^4 x\,dx \quad \left| \begin{array}{l} u = \cos x \\ du = -\sin x\,dx \\ \sin^2 x = 1 - \cos^2 x \end{array} \right|$

$= -\int (1-u^2)u^4\,du = -\int (u^4 - u^6)\,du = -\frac{u^5}{5} + \frac{u^7}{7} + C = \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{x} + C}$

(e) $\int_0^{\pi/8} \sin^2(2)\cos^3(2x)\,dx = \int_0^{\pi/8} \underbrace{\sin^2(2x)}_{1-\cos^2(2x)} \cos(2x)\cos^2(2x)\,dx \quad \left| \begin{array}{l} u = \sin(2x) \\ du = 2\cos(2x)\,dx \\ 0 \to \sin 0 = 0 \\ \frac{\pi}{8} \to \sin\frac{2\pi}{8} = \frac{\sqrt{2}}{2} \end{array} \right|$

$= \frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}} (1-u^2)u^2\,du = \frac{1}{2}\int_0^{\sqrt{2}/2} (u^2 - u^4)\,du = \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^{\sqrt{2}/2}$

$= \frac{1}{2}\left(\frac{2\sqrt{2}}{3\cdot 8} - \frac{4\cdot\sqrt{2}}{5\cdot 32}\right) = \boxed{\frac{1}{2}\left(\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{40}\right)}$

(f) $\int \sin^2 x \cos^4 x\,dx = \int \underbrace{(\sin^2 x \cos^2 x)}_{\frac{1}{4}\sin^2(2x)} \underbrace{\cos^2 x}_{\frac{1+\cos(2x)}{2}}\,dx$

$= \frac{1}{8}\int \sin^2(2x)\left(1 + \cos(2x)\right)dx = \frac{1}{8}\int \sin^2(2x)\,dx + \frac{1}{8}\int \sin^2(2x)\cos(2x)\,dx$

$$= \frac{1}{8}\int \frac{1-\cos(4x)}{2}\,dx + \frac{1}{8}\int \sin^2(2x)\cos(2x)\,dx \quad \left| \begin{array}{l} u = \sin(2x) \\ du = 2\cos(2x)dx \end{array} \right|$$

$$= \frac{1}{16}\left(x - \frac{1}{4}\sin(4x)\right) + \frac{1}{16}\int u^2\,du = \frac{1}{16}\left(x - \frac{1}{4}\sin(4x)\right) + \frac{1}{16}\,\frac{u^3}{3} + C$$

$$= \boxed{\frac{1}{16}\left(x - \frac{1}{4}\sin(4x)\right) + \frac{1}{48}\sin^3(2x) + C}$$

(g) $\displaystyle\int_0^{\pi/4} \tan^4 x \sec^2 x\,dx = \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x\,dx \\ 0 \to \tan 0 = 0 \\ \frac{\pi}{4} \to \tan\frac{\pi}{4} = 1 \end{array} \right| = \int_0^1 u^4\,du = \frac{u^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}$

(h) $\displaystyle\int \tan x \sec^3 x\,dx = \int \tan x \sec x \sec^2 x\,dx = \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x\,dx \end{array} \right|$

$$= \int u^2\,du = \frac{u^3}{3} = \boxed{\frac{\sec^3 x}{3} + C}$$

(i) $\displaystyle\int \sin(3x)\cos x\,dx$

$$\sin(3x)\cos x = \frac{1}{2}\left(\sin(3x-x) + \sin(3x+x)\right) = \frac{1}{2}\left(\sin(2x) + \sin(4x)\right)$$

$$= \frac{1}{2}\int(\sin 2x + \sin 4x)\,dx = \boxed{\frac{1}{2}\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) + C}$$

(j) $\displaystyle\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx = \int_0^{2/3} x^3\sqrt{9\left(\frac{4}{9}-x^2\right)}\,dx = 3\int_0^{2/3} x^3\sqrt{\frac{4}{9}-x^2}\,dx$

$x = \frac{2}{3}\sin t, \quad t = \sin^{-1}\left(\frac{3x}{2}\right)$

$dx = \frac{2}{3}\cos t$

$\sqrt{\frac{4}{9}-x^2} = \sqrt{\frac{4}{9}-\frac{4}{9}\sin^2 t}$

$= \frac{2}{3}\cos t$

$0 \to \sin^{-1}(0) = 0, \quad \frac{2}{3} \to \sin^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) = \sin^{-1}(1) = \pi/2$

$$= 3\int_0^{\pi/2} \frac{8}{27}\sin^3 t \; \frac{2}{3}\cos t \; \frac{2}{3}\cos t \, dt = \frac{32}{81}\int_0^{\pi/2} \sin^3 t \cos^2 t \, dt$$

$$\left| \begin{array}{l} u = \cos t \\ du = -\sin t \, dt \\ 0 \to \cos 0 = 1 \\ \frac{\pi}{2} \to \cos\frac{\pi}{2} = 0 \end{array} \right| = -\frac{32}{81}\int_1^0 (1-u^2)u^2 \, du = \frac{32}{81}\int_0^1 (u^2 - u^4)\, du$$

$$= \frac{32}{81}\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1 = \boxed{\frac{32}{81}\left(\frac{1}{3} - \frac{1}{5}\right)}$$

(k) $\displaystyle\int \frac{dx}{\sqrt{9x^2+6x-8}}$

$$9x^2+6x-8 = (9x^2+6x+1)-9 = (3x+1)^2-9$$

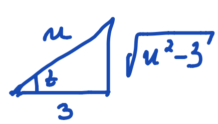
$$= \int \frac{dx}{\sqrt{(3x+1)^2-9}} \quad \left| \begin{array}{l} u = 3x+1 \\ du = 3dx \end{array} \right| = \frac{1}{3}\int \frac{du}{\sqrt{u^2-9}} \quad \left| \begin{array}{l} u = 3\sec t \\ du = 3\sec t \tan t \, dt \\ \sqrt{u^2-9} = \sqrt{9\sec^2 t - 9} = 3\tan t \end{array} \right|$$

$$= \frac{1}{3}\int \frac{3\sec t \tan t \, dt}{3\tan t} = \frac{1}{3}\int \sec t \, dt$$

$$= \frac{1}{3}\ln|\sec t + \tan t| + C$$

$\sec t = \frac{u}{3}$

$\cos t = \frac{3}{u}$



$\tan t = \frac{\sqrt{u^2-3}}{3}$

$$= \frac{1}{3}\ln\left|\frac{u}{3} + \frac{\sqrt{u^2-3}}{3}\right| + C = \boxed{\frac{1}{3}\ln\left|\frac{3x+1}{3} + \frac{\sqrt{9x^2+6x-8}}{3}\right| + C}$$

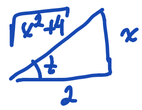
(l) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\, dx \quad \left| \begin{array}{l} x = 2\tan t \\ dx = 2\sec^2 t \, dt \\ \sqrt{x^2+4} = \sqrt{4\tan^2 t + 4} \\ = 2\sec t \end{array} \right| = \int \frac{8\tan^3 t \;\, 2\sec^2 t \, dt}{2\sec t}$

$$= 8\int \tan^3 t \sec t \, dt = 8\int \tan^2 t \, (\tan t \sec t)\, dt \quad \left| \begin{array}{l} u = \sec t \\ du = \tan t \sec t \, dt \\ \tan^2 t = \sec^2 t + 1 = u^2+1 \end{array} \right|$$

$$= 8\int (u^2+1)\,du = 8\left(\frac{u^3}{3}+u\right)+C = 8\left(\frac{\sec^3 t}{3}+\sec t\right)+C$$

$x = 2\tan t$

$\tan t = \frac{x}{2}$



$$\cos t = \frac{2}{\sqrt{x^2+4}}, \quad \sec t = \frac{\sqrt{x^2+4}}{2}$$

$$= \boxed{8\left(\frac{(x^2+4)^{3/2}}{8} + \frac{\sqrt{x^2+4}}{2}\right)+C}$$