

1. Evaluate the integral

$$(a) \int \frac{x^2 dx}{(x-3)(x+2)^2}$$

Partial fractions:

$$\begin{aligned} \frac{x^2}{(x-3)(x+2)^2} &= \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \\ &= \frac{A(x+2)^2 + B(x-3)(x+2) + C(x-3)}{(x-3)(x+2)^2} \end{aligned}$$

$$x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$$

$$x=0: 0 = 4A - 6B - 3C$$

$$x=-2: 4 = -5C \Rightarrow$$

$$C = -4/5$$

$$x=3: 9 = 25A \Rightarrow$$

$$A = 9/25$$

$$6B = 4A - 3C = \frac{36}{25} + \frac{12}{5} = \frac{96}{25}$$

$$B = \frac{16}{25}$$

$$\int \frac{x^2 dx}{(x-3)(x+2)^2} = \int \left[\frac{9}{25} \frac{1}{x-3} + \frac{16}{25} \frac{1}{x+2} - \frac{4}{5} \frac{1}{(x+2)^2} \right] dx$$

$$= \left[\frac{9}{25} \ln|x-3| + \frac{16}{25} \ln|x+2| + \frac{4}{5} \frac{1}{x+2} + C \right]$$

$$(b) \int \frac{x^4 dx}{x^4 - 1} = \int \frac{(x^4 - 1) + 1}{x^4 - 1} dx = \int \left[\frac{x^4 - 1}{x^4 - 1} + \frac{1}{x^4 - 1} \right] dx = x + \int \frac{dx}{x^4 - 1}$$

Partial fractions:

$$\frac{1}{x^4 - 1} = \frac{1}{(x^2 - 1)(x^2 + 1)} = \frac{1}{(x - 1)(x + 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{Cx + D}{x^2 + 1}$$

$$= \frac{A(x + 1)(x^2 + 1) + B(x - 1)(x^2 + 1) + (Cx + D)(x^2 - 1)}{x^4 - 1}$$

$$1 = A(x + 1)(x^2 + 1) + B(x - 1)(x^2 + 1) + (Cx + D)(x^2 - 1)$$

$$x = 1: 1 = 4A \Rightarrow \boxed{A = 1/4}$$

$$x = -1: 1 = -4B \Rightarrow \boxed{B = -1/4}$$

$$x = 0: 1 = A - B - D \Rightarrow D = A - B - 1 = \frac{1}{4} + \frac{1}{4} - 1 = \boxed{\frac{1}{2} = D}$$

$$x = 2: 1 = A(3)(5) + B(1)(5) + (2C + D)(4 - 1)$$

$$1 = 15A + 5B + 6C + 3D$$

$$6C = 1 - 15A - 5B - 3D = 1 - \frac{15}{4} + \frac{5}{4} - \frac{3}{2} = \frac{4 - 10 - 6}{4} = -\frac{12}{4} = -3$$

$$\boxed{C = -1/2}$$

$$\int \frac{dx}{x^4 - 1} = \int \left[\frac{1}{4} \frac{1}{x - 1} - \frac{1}{4} \frac{1}{x + 1} + \frac{1}{2} \frac{-x + 1}{x^2 + 1} \right] dx$$

$$= \frac{1}{4} \ln|x - 1| - \frac{1}{4} \ln|x + 1| - \frac{1}{4} \int \frac{2x dx}{x^2 + 1} + \frac{1}{2} \int \frac{dx}{x^2 + 1}$$

$u = x^2 + 1$
 $du = 2x dx$

$$= \frac{1}{4} \ln|x - 1| - \frac{1}{4} \ln|x + 1| - \frac{1}{4} \int \frac{du}{u} + \frac{1}{2} \tan^{-1} x$$

$$= \frac{1}{4} \ln|x - 1| - \frac{1}{4} \ln|x + 1| - \frac{1}{4} \ln|u| + \frac{1}{2} \tan^{-1} x + C$$

$$= \boxed{\frac{1}{4} \ln|x - 1| - \frac{1}{4} \ln|x + 1| - \frac{1}{4} \ln|x^2 + 1| + \frac{1}{2} \tan^{-1} x + C}$$

$$(c) \int \frac{x^4 + 1}{x(x^2 + 1)^2} dx$$

Partial fractions:

$$\frac{x^4 + 1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + F}{(x^2 + 1)^2}$$

$$\frac{x^4 + 1}{x(x^2 + 1)^2} = \frac{A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + F)x}{x(x^2 + 1)^2}$$

$$x^4 + 1 = A(x^4 + 2x^2 + 1) + (Bx + C)(x^3 + x) + Dx^2 + Fx$$

$$x^4 + 1 = x^4(A + B) + x^3(C) + x^2(2A + B + D) + x(C + F) + A$$

$$x^4: 1 = A + B$$

$$x^3: 0 = C$$

$$x^2: 0 = 2A + B + D$$

$$x^1: 0 = C + F$$

$$x^0: 1 = A$$

$$C = 0, F = 0, B = 0$$

$$D = -2A = -2$$

$$A = 1$$

$$= \int \left[\frac{1}{x} - \frac{2x}{(x^2 + 1)^2} \right] dx = \ln|x| - \int \frac{2x}{(x^2 + 1)^2} dx \quad \left| \begin{array}{l} u = x^2 + 1 \\ du = 2x dx \end{array} \right|$$

$$= \ln|x| - \int \frac{du}{u^2} = \ln|x| + \frac{1}{u} + C$$

$$= \ln|x| + \frac{1}{x^2 + 1} + C$$

$$(d) \int \frac{x dx}{x^2 + x + 1} = \int \frac{x dx}{\underbrace{\left(x^2 + \frac{1}{2}(2)x + \frac{1}{4}\right)}_{\left(x + \frac{1}{2}\right)^2} + \frac{3}{4}}$$

$$\begin{aligned} a=x \quad \begin{matrix} a^2 & 2ab \\ \parallel & \parallel \\ \boxed{x^2} & + \boxed{x} + 1 \end{matrix} &= x^2 + 2 \cdot x \cdot \underbrace{\left(\frac{1}{2}\right)}_{\parallel \atop b} + 1 = \underbrace{x^2 + 2x\left(\frac{1}{2}\right)}_{\left(x + \frac{1}{2}\right)^2} + \underbrace{\frac{1}{4} - \frac{1}{4} + 1}_{\frac{3}{4}} \\ (a+b)^2 &= a^2 + 2ab + b^2 \end{aligned}$$

$$= \int \frac{x dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \quad \left| \begin{array}{l} u = x + \frac{1}{2} \\ du = dx \\ x = u - \frac{1}{2} \end{array} \right| = \int \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}} du$$

$$= \frac{1}{2} \int \frac{2u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \int \frac{du}{u^2 + \frac{3}{4}}$$

$$\begin{aligned} \nearrow \\ u^2 + \frac{3}{4} = v &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \\ dv &= 2u du \end{aligned}$$

$$= \frac{1}{2} \int \frac{dv}{v} - \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{3}{4}}} \cdot \tan^{-1} \left(\frac{u}{\frac{\sqrt{3}}{2}} \right) + C$$

$$= \frac{1}{2} \ln|v| - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(x + \frac{1}{2}\right) \right) + C$$

$$= \boxed{\frac{1}{2} \ln \left| \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \right| - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(x + \frac{1}{2}\right) \right) + C}$$

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C}$$

2. Determine whether the integral is convergent. Evaluate those that are convergent.

$$(a) \int_0^{\infty} \frac{dx}{(x+2)(x+3)} = \lim_{a \rightarrow \infty} \int_0^a \frac{dx}{(x+2)(x+3)}$$

$$\frac{1}{(x+2)(x+3)} \sim \frac{1}{x^2}, \quad p=2 > 1 - \text{conv.}$$

Partial fractions:

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} = \frac{A(x+3) + B(x+2)}{(x+3)(x+2)}$$

$$1 = A(x+3) + B(x+2)$$

$$x = -3: \quad 1 = -B \Rightarrow \boxed{B = -1}$$

$$x = -2: \quad \boxed{1 = A}$$

$$\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$$

$$\lim_{a \rightarrow \infty} \int_0^a \frac{dx}{(x+2)(x+3)} = \lim_{a \rightarrow \infty} \int_0^a \left[\frac{1}{x+2} - \frac{1}{x+3} \right] dx$$

$$= \lim_{a \rightarrow \infty} \left[\ln|x+2| - \ln|x+3| \right]_0^a = \lim_{a \rightarrow \infty} \ln \left| \frac{x+2}{x+3} \right| \Big|_0^a$$

$$= \lim_{a \rightarrow \infty} \ln \left| \frac{a+2}{a+3} \right| - \ln \frac{2}{3} = \ln \left(\underbrace{\lim_{a \rightarrow \infty} \frac{a+2}{a+3}}_1 \right) - \ln \frac{2}{3}$$

$$= \ln 1 - \ln \frac{2}{3} = \boxed{-\ln \frac{2}{3}} = \boxed{\ln \frac{3}{2}}$$

$$(b) \int_0^{\infty} x e^{-x} dx = \lim_{a \rightarrow \infty} \int_0^a x e^{-x} dx \quad \left| \begin{array}{ll} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{array} \right|$$

$$= \lim_{a \rightarrow \infty} \left(-x e^{-x} \right]_0^a + \int_0^a e^{-x} dx \right)$$

$$= \lim_{a \rightarrow \infty} \left(-a e^{-a} - e^{-x} \right]_0^a \right) = -\lim_{a \rightarrow \infty} a e^{-a} - \overbrace{\lim_{a \rightarrow \infty} e^{-a}}^0 + 1$$

$$= -\lim_{a \rightarrow \infty} \frac{a}{e^a} + 1 = -\underbrace{\lim_{a \rightarrow \infty} \frac{1}{e^a}}_0 + 1 = \boxed{1}$$

$$(c) \int_1^{17} \frac{dx}{\sqrt[3]{x-9}} = \int_1^9 \frac{dx}{\sqrt[3]{x-9}} + \int_9^{17} \frac{dx}{\sqrt[3]{x-9}}$$

$$\frac{1}{\sqrt[3]{x-9}} \sim \frac{1}{x^{1/3}}, \quad p = 1/3 < 1 - \text{conv.}$$

$$= \lim_{a \rightarrow 9^-} \int_1^a \frac{dx}{(x-9)^{1/3}} + \lim_{b \rightarrow 9^+} \int_b^{17} \frac{dx}{(x-9)^{1/3}}$$

$$= \lim_{a \rightarrow 9^-} \left(\frac{(x-9)^{-1/3+1}}{-1/3+1} \right) \Big|_1^a + \lim_{b \rightarrow 9^+} \left(\frac{3}{2} (x-9)^{2/3} \right) \Big|_b^{17}$$

$$= \lim_{a \rightarrow 9^-} \left(\frac{3}{2} (a-9)^{2/3} - \frac{3}{2} (1-9)^{2/3} \right) + \lim_{b \rightarrow 9^+} \left(\frac{3}{2} (17-9)^{2/3} - \frac{3}{2} (b-9)^{2/3} \right)$$

$$= \underbrace{\frac{3}{2} \lim_{a \rightarrow 9^-} (a-9)^{2/3}}_0 - \frac{3}{2} (-8)^{2/3} + \frac{3}{2} 8^{2/3} - \underbrace{\frac{3}{2} \lim_{b \rightarrow 9^+} (b-9)^{2/3}}_0$$

$$= -\frac{3}{2} (4) + \frac{3}{2} (4) = \boxed{0}$$

3. Find the length of the curve.

(a) $y = \ln(\sin x)$, $\pi/6 \leq x \leq \pi/3$

$$L = \int_{\pi/6}^{\pi/3} \sqrt{1 + [y'(x)]^2} dx = \int_{\pi/6}^{\pi/3} \sqrt{1 + \left[\frac{\cos x}{\sin x}\right]^2} dx$$

$$= \int_{\pi/6}^{\pi/3} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx = \int_{\pi/6}^{\pi/3} \frac{1}{\sin x} dx = \int_{\pi/6}^{\pi/3} \csc x dx$$

$$= [\ln(\csc x - \cot x)]_{\pi/6}^{\pi/3} = \boxed{\ln\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) - \ln(2 - \sqrt{3})}$$

(b) $x = y^{3/2}$, $0 \leq y \leq 1$

$$\begin{aligned}
 L &= \int_0^1 \sqrt{1 + [x'(y)]^2} dy = \int_0^1 \sqrt{1 + \left(\frac{3}{2} y^{1/2}\right)^2} dy = \int_0^1 \sqrt{1 + \frac{9}{4} y} dy \quad \left\{ \begin{array}{l} u = 1 + \frac{9}{4} y \\ du = \frac{9}{4} dy \\ 0 \rightarrow 1 \\ 1 \rightarrow 1 + \frac{9}{4} \\ \quad = \frac{13}{4} \end{array} \right. \\
 &= \frac{4}{9} \int_1^{13/4} \sqrt{u} du = \frac{4}{9} \left[\frac{u^{3/2}}{3/2} \right]_1^{13/4} \\
 &= \frac{8}{27} \left[\left(\frac{13}{4}\right)^{3/2} - 1 \right] = \boxed{\frac{8}{27} \left(\frac{13\sqrt{13}}{8} - 1 \right)}
 \end{aligned}$$

(c) $x = 3t - t^3, y = 3t^2, 0 \leq t \leq 2$

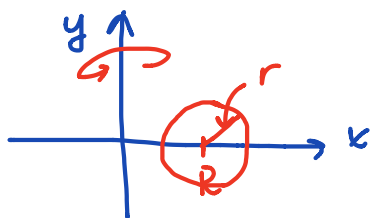
$$L = \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_0^2 \sqrt{(3-3t^2)^2 + (6t)^2} dt$$

$$= \int_0^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2} dt = \int_0^2 \sqrt{9 + 18t^2 + 9t^4} dt$$

$$= \int_0^2 \sqrt{(3+3t^2)^2} dt = \int_0^2 (3+3t^2) dt = \left[3t + \frac{3t^3}{3} \right]_0^2$$

$$= 6 + 8 = \boxed{14}$$

4. Find the surface area of a torus.



The torus is generated by rotating the circle $(x-R)^2 + y^2 = r^2$ ($R > r$) about the y -axis.

We do the surface area of the surface generated by the upper half of the circle, and multiply it by 2.

The equation of the upper semicircle is

$$y = \sqrt{r^2 - (x-R)^2}, \quad R-r \leq x \leq R+r$$

$$y' = \frac{2(x-R)}{2\sqrt{r^2 - (x-R)^2}} = \frac{x-R}{\sqrt{r^2 - (x-R)^2}}$$

$$S.A. = 4\pi \int_{R-r}^{R+r} x \sqrt{1 + (y')^2} dx = 4\pi \int_{R-r}^{R+r} x \sqrt{1 + \frac{(x-R)^2}{r^2 - (x-R)^2}} dx$$

$$= 4\pi \int_{R-r}^{R+r} x \sqrt{\frac{r^2 - (x-R)^2 + (x-R)^2}{r^2 - (x-R)^2}} dx = 4\pi \int_{R-r}^{R+r} x \frac{r}{\sqrt{r^2 - (x-R)^2}} dx$$

$$= 4\pi \int_{-r}^r \frac{(u+R)r}{\sqrt{r^2 - u^2}} du = 4\pi r \left(\int_{-r}^r \frac{u du}{\sqrt{r^2 - u^2}} + R \int_{-r}^r \frac{du}{\sqrt{r^2 - u^2}} \right)$$

$v = r^2 - u^2$
 $-r \rightarrow 0, r \rightarrow 0$

$$\begin{array}{l} x = u + R \\ u = x - R \\ du = dx \\ R-r \rightarrow \\ R-r-R = -r \\ R+r \rightarrow \\ R+r-R = r \end{array}$$

$$= 4\pi Rr \sin^{-1}\left(\frac{r}{u}\right) \Big|_{-r}^r = 4\pi Rr [\sin^{-1}(1) - \sin^{-1}(-1)] = 4\pi Rr \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \boxed{4\pi^2 Rr}$$

5. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

(a) $a_n = \sin n$ *divergent*

$\lim_{n \rightarrow \infty} \sin n$ DNE

(a) $u_n = \sin n$

(b) $a_n = \frac{n}{\ln n}$ divergent

$$\lim_{n \rightarrow \infty} \frac{n}{\ln n} = \frac{\infty}{\infty} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n}} = \lim_{n \rightarrow \infty} n = \infty$$

(c) $a_n = \frac{\pi^n}{3^n}$ diverges

$$\frac{\pi}{3} > 1, \lim_{n \rightarrow \infty} \left(\frac{\pi}{3}\right)^n = \infty$$

(d) $a_n = \frac{n}{2n+5}$ converges to $\frac{1}{2}$

$$\lim_{n \rightarrow \infty} \frac{n}{2n+5} = \frac{1}{2}$$

(e) $a_n = \sqrt{n+2} - \sqrt{n-1}$ converges to 0

$$\lim_{n \rightarrow \infty} (\sqrt{n+2} - \sqrt{n-1}) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n+2} + \sqrt{n-1})(\sqrt{n+2} - \sqrt{n-1})}{\sqrt{n+2} + \sqrt{n-1}}$$

$$= \lim_{n \rightarrow \infty} \frac{n+2 - (n-1)}{\sqrt{n+2} + \sqrt{n-1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+2} + \sqrt{n-1}} = 0$$

6. Show that the sequence defined by $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$ is increasing and $a_n < 3$ for all n . Find its limit.

$$a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}$$

show that $\{a_n\}$ is increasing.
math induction.

Step 1. $a_1 = 1, \quad a_2 = 3 - \frac{1}{a_1} = 3 - 1 = 2 \quad a_1 < a_2$

$$a_3 = 3 - \frac{1}{a_2} = 3 - \frac{1}{2} = \frac{5}{2} > 2 \quad a_2 < a_3$$

Step 2. $a_1 < a_2 < \dots < a_{n-1} < a_n$

Step 3. show that $a_{n+1} > a_n$

$$a_{n+1} = 3 - \frac{1}{a_n}, \quad a_n = 3 - \frac{1}{a_{n-1}}$$

$$a_{n-1} < a_n$$

$$(-1) \frac{1}{a_{n-1}} > \frac{1}{a_n} \quad (-1)$$

$$3 + -\frac{1}{a_{n-1}} < -\frac{1}{a_n} + 3$$

$$\underbrace{3 - \frac{1}{a_{n-1}}}_{a_n} < \underbrace{3 - \frac{1}{a_n}}_{a_{n+1}}$$

$$a_n < a_{n+1}$$

Conclusion: $a_1 < a_2 < a_3 < \dots < a_n < \dots$ (the sequence is increasing for all n).

show that $a_n < 3$.

$$a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}$$

$$a_n > a_1 > 1 \quad (\text{the sequence is increasing})$$

$$a_n > 1, \text{ then } \frac{1}{a_n} < 1$$

$$3 - \frac{1}{a_n} < 3$$

$$\underline{\underline{L}} = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(3 - \frac{1}{a_{n-1}} \right) = \lim_{n \rightarrow \infty} 3 - \lim_{n \rightarrow \infty} \frac{1}{a_{n-1}} \quad 1 < L < 3.$$

$$= 3 - \frac{1}{\lim_{n \rightarrow \infty} a_{n-1}}$$

$$\boxed{\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n-1}}$$

$$= 3 - \frac{1}{L}$$

$$L = 3 - \frac{1}{L} \quad (L \neq 0)$$

$$L^2 - 3L + 1 = 0$$

$$L_1 = \frac{3 + \sqrt{9-4}}{2} = \frac{3 + \sqrt{5}}{2} \approx 2.62, \quad L_2 = \frac{3 - \sqrt{5}}{2} \approx 0.38$$

$$\boxed{\lim_{n \rightarrow \infty} a_n = \frac{3 + \sqrt{5}}{2}}$$