7. Find the sum of the series

(a) $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^{3n-1}}$

$$= \sum_{n=1}^{\infty} \frac{2 \cdot 2^{2n}}{\frac{1}{3} 3^{3n}}$$

$$= \sum_{n=1}^{\infty} 6 \left(\frac{2^2}{3^3}\right)^n$$

$$= \sum_{n=1}^{\infty} 6 \left(\frac{4}{27}\right)^n$$

$$= \sum_{n=1}^{\infty} 6 \left(\frac{4}{27}\right) \left(\frac{4}{27}\right)^{n-1}$$

$$= \frac{6 \frac{4}{27}}{1 - \frac{4}{27}}$$

$$= \frac{\frac{24}{27}}{\frac{23}{27}}$$

$$= \boxed{\frac{24}{23}}$$

(b) $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ Partial fractions:

$$\frac{1}{n^2-4} = \frac{1}{(n-2)(n+2)} = \frac{A}{n-2} + \frac{B}{n+2}$$

$$= \frac{A(n+2) + B(n-2)}{(n-2)(n+2)}$$

$$1 = A(n+2) + B(n-2)$$

$n=-2$: $1 = -4B \longrightarrow B = -\frac{1}{4}$

$n=2$: $1 = 4A \longrightarrow A = \frac{1}{4}$

$$\frac{1}{n^2-4} = \frac{1}{4}\left(\frac{1}{n-2} - \frac{1}{n+2}\right)$$

Partial sums:

$$S_3 = \frac{1}{4}\left(\frac{1}{1} - \frac{1}{5}\right) = a_3$$

$$a_3 + a_4 = S_4 = \frac{1}{4}\left(\frac{1}{1} - \frac{1}{5}\right) + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right)$$

$$S_5 = a_3 + a_4 + a_5 = \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right) + \frac{1}{4}\left(\frac{1}{3} - \frac{1}{7}\right)$$

$$S_6 = a_3 + a_4 + a_5 + a_6 = \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7}\right)$$

$$+ \frac{1}{4}\left(\frac{1}{4} - \frac{1}{8}\right)$$

$$S_7 = a_3 + a_4 + a_5 + a_6 + a_7 = \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \cancel{\frac{1}{5}} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8}\right)$$

$$+ \frac{1}{4}\left(\cancel{\frac{1}{5}} - \frac{1}{9}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9}\right)$$

$$S_n = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$S = \lim_{n\to\infty} S_n = \frac{1}{4} \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{4} \frac{24+12+8+6}{24}$$

$$= \frac{1}{4} \frac{50}{24}$$

$$= \boxed{\frac{25}{48}}$$

1. Which of the following series is convergent?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7}+1}$

compare with $\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7}} = \sum_{n=1}^{\infty} n^{2-5/7} = \sum_{n=1}^{\infty} n^{9/7}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{9/7} = \infty$

Divergent by the Test for Divergence ($\lim_{n\to\infty} a_n \neq 0$)

(b) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$ $\qquad 0 \leq \cos^2 n \leq 1$

$\frac{\cos^2 n}{3^n} \leq \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric series, $r = \frac{1}{3} < 1$)

By Comparison Test 1, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$ converges.

∞ 1

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Do the Integral Test.

$f(x) = \frac{1}{x(\ln x)^2}$ is positive on $[2, \infty)$

$x(\ln x)^2$ turns zero at $x=0$, $x=1$.

$\frac{1}{x(\ln x)^2}$ is continuous on $[2,\infty)$.

$f'(x) = -1(x(\ln x))^{-2}\left[(\ln x)^2 - 2x \ln x \cdot \frac{1}{x}\right]$

$= -1\left(\frac{\ln^2 x - 2}{x^2(\ln x)^4}\right) = \frac{2 - \ln^2 x}{x^2 \ln^4 x} < 0$

$2 - \ln^2 x < 0$, $\ln^2 x > 2$, $\ln x > \sqrt{2}$

$x > e^{\sqrt{2}} \approx 4.06$

$f(x)$ is decreasing on $[4.06, \infty)$

Can do the Integral Test.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \left| \begin{matrix} u = \ln x \\ du = \frac{dx}{x} \end{matrix} \right| = \int_{\ln 2}^{\infty} \frac{du}{u^2} = -\frac{1}{u}\Big|_{\ln 2}^{\infty} = 0 - \frac{1}{\ln 2}$$

Since $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \frac{1}{\ln 2}$ convergent,

then, by the Integral Test,

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

4. Which of the following series is absolutely convergent?

(a) $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ Ratio Test for $a_n = \frac{(-3)^n}{n!}$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(-3)}{n+1} \right| = 0 < 1$

converges absolutely

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ - diverges (harmonic series).

$b_n = \frac{1}{n}$ $\quad b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges by AST, but not absolutely converges.

(c) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{\sqrt{n-2}}$

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n}{\sqrt{n-2}}\right| = \sum_{n=1}^{\infty}\frac{n}{\sqrt{n-2}}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \lim_{n\to\infty}\frac{n}{\sqrt{n}\sqrt{1-\frac{2}{n}}} =$$

$$= \lim_{n\to\infty}\sqrt{n} = \infty.$$

diverges by the Test ~~by~~ for Divergence.

AST: $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \infty.$

diverges by AST.

$\boxed{\text{The series diverges.}}$

(d) $\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{3^{3n}}$

$$\sum_{n=0}^{\infty}\left|(-1)^n\frac{2^{2n}}{3^{3n}}\right| = \sum_{n=0}^{\infty}\frac{2^{2n}}{3^{3n}} = \sum_{n=0}^{\infty}\left(\frac{4}{27}\right)^n$$

converges (geometric series for $r=\frac{4}{27}<1$).

The series converges absolutely.

5. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$.

The radius of converges

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|, \text{ where } c_n = \frac{2^n}{\sqrt{n+3}}$$

$$R = \lim_{n\to\infty} \left|\frac{2^n}{\sqrt{n+3}} \cdot \frac{\sqrt{n+4}}{2^{n+1}}\right| = \frac{1}{2}.$$

The interval of convergence:

$$|x-3| < \frac{1}{2}$$

$$-\frac{1}{2} < x-3 < \frac{1}{2}$$

$$+\frac{5}{2} < x < \frac{7}{2}$$

End points: $x = +\frac{5}{2} \to \sum_{n=1}^{\infty} \frac{2^n\left(+\frac{5}{2}-3\right)^n}{\sqrt{n+3}}$

$$= \sum_{n=1}^{\infty} \cancel{(-1)^n} \frac{2^n\left(-\frac{1}{2}\right)^n}{\sqrt{n+3}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}} \text{ - converges but not absolutely.}$$

$x = \frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{7}{2}-3\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}$ diverges.

interval of convergence: $\boxed{\left[\frac{5}{2}, \frac{7}{2}\right)}$.

$\boxed{R = \frac{1}{2}}$

4

6. Find the power series representation for the function $f(x) = \ln(1-2x)$ centered at 0.

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (+1)^n (2x)^n = \sum_{n=0}^{\infty} (+1)^n 2^n x^n$$

$$\ln(1-2x) = -2 \int \frac{1}{1-2x} dx = -2 \int \left(\sum_{n=0}^{\infty} (+1)^n 2^n x^n \right) dx$$

$$= -2 \sum_{n=0}^{\infty} (+1)^n 2^n \left(\int x^n dx \right) = \sum_{n=0}^{\infty} -1(+2)^{n+1} \frac{x^{n+1}}{n+1} + C$$

C: plug $x=0$;

$\ln 1 = C \Rightarrow C = 0.$

$$\boxed{\ln(1-2x) = \sum_{n=0}^{\infty} (-1)(+2)^{n+1} \frac{x^{n+1}}{n+1}}$$

7. Find the Taylor series for $f(x) = xe^{2x}$ at $x = 2$.

$f(x) = xe^{2x}$

$f'(x) = e^{2x} + 2xe^{2x} = 1 \cdot 2^0 e^{2x} + 2^1 xe^{2x}$

$f''(x) = 2e^{2x} + 2e^{2x} + 4xe^{2x}$

$= 4e^{2x} + 4xe^{2x} = 2 \cdot 2^1 e^{2x} + 2^2 xe^{2x}$

$f'''(x) = 8e^{2x} + 4e^{2x} + 8xe^{2x}$

$= 12e^{2x} + 8xe^{2x} = 3 \cdot 2^2 e^{2x} + 2^3 xe^{2x}$

$f^{(n)}(x) = n \cdot 2^{n-1} e^{2x} + 2^n x e^{2x}$

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{(n2^{n-1} + 2^n \cdot 2)e^4}{n!} (x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(n2^{n-1} + 2^{n+1})e^4}{n!} (x-2)^n$$

8. Find the Maclaurin series for $f(x) = x\sin(x^3)$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$$

$$x\sin x^3 = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}$$

9. Find the sum of the series

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n x^2}{n!}$ $\displaystyle = x^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} = x^2\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} - 1 + 1\right]$

$$= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = x^2 e^{-1}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$ $\displaystyle = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{6}\right)^{2n} \frac{1}{(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

10. Evaluate the indefinite integral as a power series $\int e^{x^2}\,dx$.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\int e^{x^2} dx = \int\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right) dx = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int x^{2n} dx\right)$$

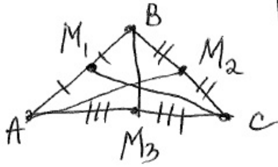
$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\,n!} + C$$

10. Find the length of the medians of the triangle withvertices $A(1,2,3)$, $B(-2,0,5)$, $C(4,1,5)$.

11. Find the equation of the sphere with center $(2,-3,6)$ that touches the yz-plane.

12. Find an eqution of the set of all points equidistant from the points $A(-1,5,3)$ and $B(6,2,-2)$.

#10.



$A(1,2,3)$, $B(-2,0,5)$, $C(4,1,5)$

midpoints for the sides:

$$M_1\left(\frac{1+(-2)}{2}, \frac{2+0}{2}, \frac{3+5}{2}\right) = M_1\left(-\frac{1}{2}, 1, 4\right)$$

$$M_2\left(\frac{-2+4}{2}, \frac{0+1}{2}, \frac{5+5}{2}\right) = M_2\left(1, \frac{1}{2}, 5\right)$$

$$M_3\left(\frac{1+4}{2}, \frac{2+1}{2}, \frac{3+5}{2}\right) = M_3\left(\frac{5}{2}, \frac{3}{2}, 4\right)$$

medians:

$$\cancel{CM_1 = \sqrt{\left(-\frac{1}{2}+1\right)^2 + (1-2)^2 +}}$$

$$CM_1 = \sqrt{\left(4+\frac{1}{2}\right)^2 + (1-1)^2 + (4-5)^2} = \sqrt{\frac{81}{4}+1} = \boxed{\frac{\sqrt{85}}{2}}$$

$$AM_2 = \sqrt{(1-1)^2 + \left(\frac{1}{2}-2\right)^2 + (5-3)^2} = \sqrt{\frac{9}{4}+4} = \boxed{\frac{\sqrt{25}}{2}} = \boxed{\frac{5}{2}}$$

$$BM_3 = \sqrt{\left(\frac{5}{2}+2\right)^2 + \left(\frac{3}{2}\right)^2 + (4-5)^2} = \sqrt{\frac{81}{4}+\frac{9}{4}+1}$$

$$= \sqrt{\frac{94}{4}} = \boxed{\frac{\sqrt{94}}{2}}$$

#11. $R =$ the distance from $(2, -3, 6)$ to the (yz)-plane

$R = 2$

Equation: $\sqrt{(x-2)^2 + (\cancel{3}\,y+3)^2 + (z-6)}$

#12. $P(x, y, z)$, $A(-1, 5, 3)$, $B(6, 2, -2)$

$PA = PB$

$PA = \sqrt{(-1-x)^2 + (5-y)^2 + (3-z)^2}$

$PB = \sqrt{(6-x)^2 + (2-y)^2 + (-2-z)^2}$

$(-1-x)^2 + (5-y)^2 + (3-z)^2 = (6-x)^2 + (2-y)$

$1 + 2x + \cancel{x^2} + 25 - 15y + \cancel{y^2} + 9 - 6z + \cancel{z^2}$

$= 36 - 12x + \cancel{x^2} + 4 - 4y + \cancel{y^2} + 4 + 4z$

$2x + 12x - 15y + 4y - 6z - 4z + 35 - 44 = 0$

$\boxed{14x - \cancel{9x}\ 11y - 10z - 9 = 0}$ plane.