Section 6.5 **The Substitution Rule**

The substitution rule for indefinite integrals If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boxed{\int f(g(x))g'(x)dx = \int f(u)du} \quad u = g(x)$$

Example 1. Evaluate each integral:

1. $\frac{1}{3}\int 3x^2 e^{x^3} dx = \left| \begin{array}{l} u = x^3 \\ du = 3x^2 dx \end{array} \right| = \frac{1}{3}\int e^u du$

differential of u

$= \frac{1}{3} e^u + C$

$= \boxed{\frac{1}{3} e^{x^3} + C}$

2. $\displaystyle\int \frac{x + \arcsin x}{\sqrt{1-x^2}}dx$

3. $\displaystyle\int \sec x \tan x \sqrt{1 + \sec x}dx$

$u = \sec x + 1$
$du = \sec x \tan x \, dx$

$= \int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + C = \boxed{\frac{2}{3}(\sec x + 1)^{3/2} + C}$

The substitution rule for definite integrals If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 2. Evaluate the integral:

1. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

$u = \ln x$

$du = \dfrac{dx}{x}$

$e \longmapsto \ln e = 1$

$e^4 \longmapsto \ln(e^4) = 4$

$$= \int_1^4 \frac{du}{\sqrt{u}}$$

$$= \left.\frac{u^{1/2}}{1/2}\right]_1^4$$

$$= 2[2-1]$$

$$= \boxed{2}$$

$$2.\ \frac{1}{2}\int_0^1 \frac{\overset{du}{2x\,dx}}{\sqrt{1+x^4}} = \left| \begin{array}{c} u = x^2 \\ x^4 = u^2 \\ du = 2x\,dx \\ 0 \longmapsto 0^2 = 0 \\ 1 \longmapsto 1^2 = 1 \end{array} \right| = \frac{1}{2}\int_0^1 \frac{du}{\sqrt{1+u^2}} = \left[\frac{1}{2}\ln\left|u+\sqrt{1+u^2}\right|\right]_0^1$$

$$= \frac{1}{2}\ln\left(1+\sqrt{1+1}\right) - \frac{1}{2}\cancel{\ln 1}^{\,0}$$

$$= \boxed{\frac{1}{2}\ln\left(1+\sqrt{2}\right)}$$

If $F(x)$ is an antiderivative to $f(x)$, then

$u = ax+b$

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$$

Example 3. Evaluate

1. $\int \sin 5x dx$ $=$ $-\frac{1}{5}\cos 5x + C$

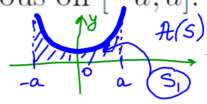
2

$\int x^{-1/2} dx = 2x^{1/2} + C$

2. $\int \frac{dx}{\sqrt{3x+1}}$ $=$ $\frac{2}{\cancel{3}}(\cancel{3}x+1)^{1/2} + C$

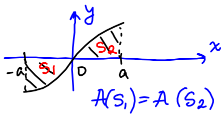
$u = 3x+1$

Integrals of symmetric functions Suppose f is continuous on $[-a, a]$.



(a) If f is **even**, $f(x) = f(-x)$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

(b) If f is **odd**, $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)dx = 0$



$\int_{-a}^{a} f(x)dx = A(S_2) - A(S_1) = 0$

Example 4. Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6} dx. = \boxed{0}$

x^2 – even

$\frac{1}{1+x^6}$ – even

$\frac{x^2}{1+x^6}$ even

$\sin x$ odd

$\frac{x^2 \sin x}{1+x^6}$ – odd