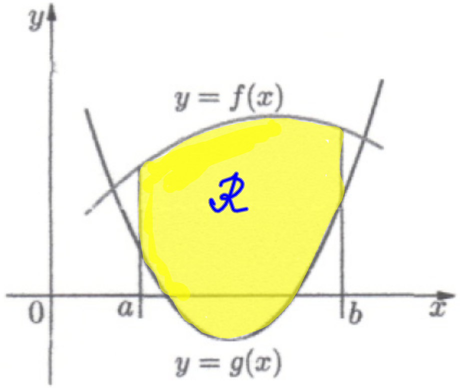
Section 7.1 **Areas between curves**

The area of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$\boxed{A(\mathcal{R}) = \int_a^b [f(x) - g(x)]dx}$$

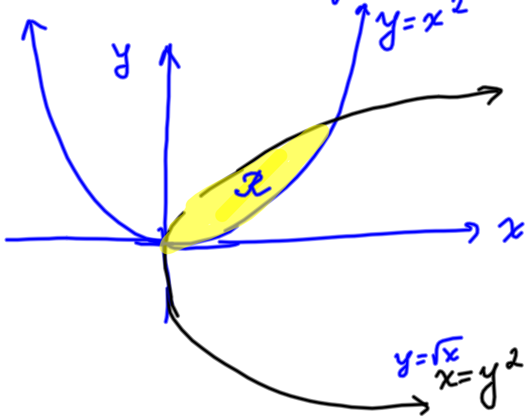


Example 1. Find the area of the region bounded by

~~Points~~

1. $y = x^2$, $y^2 = x$ ($y = \sqrt{x}$)

1) Graph the region.



$A(R)$ - ?

2) Points of intersection:

$(x^2)^2 = (\sqrt{x})^2$

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$x(x-1)(x^2 + x + 1) = 0$

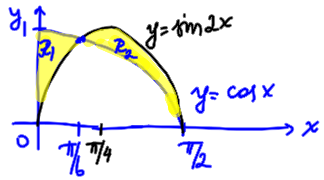
$x_1 = 0, \quad x_2 = 1$ limits of integration

3) $A(R) = \int_0^1 [\sqrt{x} - x^2]\,dx = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$

$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$

2. $y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$

1. Graph the region



2. Points of intersection on $(0, \pi/2)$

$\cos x = \sin 2x$

$\cos x = 2\sin x \cos x$

$2\sin x \cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ 　　　　 $2\sin x - 1 = 0$

$x_1 = 0, x_2 = \pi/2$ 　　　　 $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$

$$A(R_1) = \int_0^{\pi/6} [\cos x - \sin 2x]\,dx = \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\pi/6}$$

$$= \sin\frac{\pi}{6} - \sin 0 + \frac{1}{2}\cos\frac{\pi}{3} - \frac{1}{2}\cos 0$$

$$= \frac{1}{2} - 0 + \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}$$

$$= \boxed{\frac{1}{4}}$$

$$A(R_2) = \int_{\pi/6}^{\pi/2} [\sin 2x - \cos x]\,dx = \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$

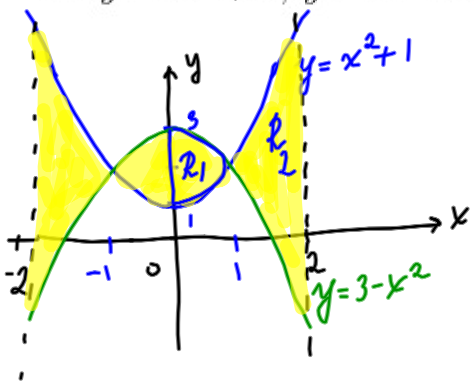
$$= -\frac{1}{2}\cos\pi - \sin\frac{\pi}{2} + \frac{1}{2}\cos\frac{\pi}{3} + \sin\frac{\pi}{6}$$

$$= \frac{1}{2} - 1 + \frac{1}{4} + \frac{1}{2}$$

$$= \boxed{\frac{1}{4}}$$

Total area $= A(R_1) + A(R_2) = \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$

3. $y = x^2 + 1$, $y = 3 - x^2$, $x = -2$, $x = 2$



Points of intersection:

$x^2+1 = 3-x^2$

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

The region is symmetric about the y-axis.

$A(R) = 2[A(R_1) + A(R_2)]$

$0 \le x \le 2$

$$A(R_1) = \int_0^1 [3-x^2-(x^2+1)]dx = \int_0^1 [2-2x^2]\,dx = \left[2x - \frac{2x^3}{3}\right]_0^1 = 2 - \frac{2}{3} = \frac{4}{3}$$

$$A(R_2) = \int_1^2 [x^2+1-[3-x^2]]\,dx = \int_1^2 [2x^2-2]\,dx = \left[\frac{2x^3}{3} - 2x\right]_1^2 = \frac{16}{3} - 4 - \frac{2}{3} + 2 = \frac{8}{3}$$

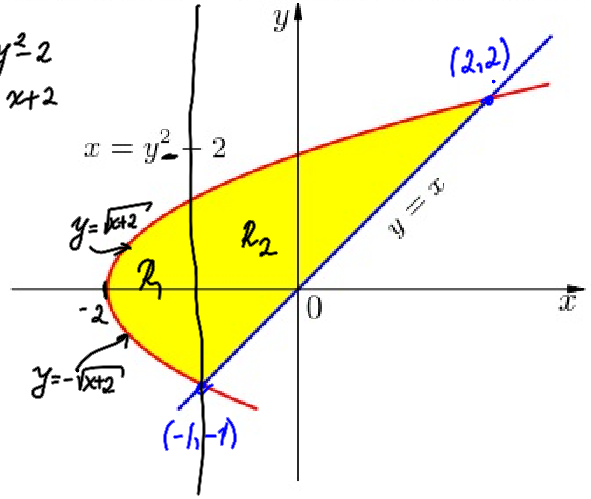
Total area $= \left[\frac{8}{3} + \frac{4}{3}\right]2 = \boxed{8}$

In general case, the area between the curves $y = f(x)$, $y = g(x)$ and between $x = a$ and $x = b$, is

$$\boxed{A = \int_a^b |f(x) - g(x)|dx}$$

Example 2. Find the area of the shaded region.

$x = y^2 - 2$
$y^2 = x + 2$



Points of intersection.

$y^2 - 2 = y$

$y^2 - y - 2 = 0$

$(y+1)(y-2) = 0$

$y_1 = -1, \quad y_2 = 2$

Corresponding x-coord.

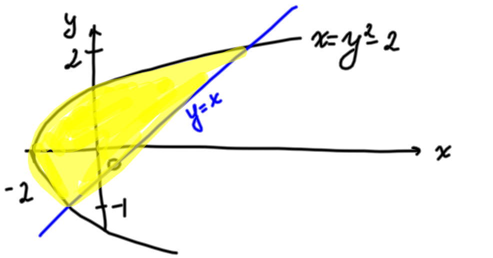
$x_1 = -1, \quad x_2 = 2.$

$$A(R_1) = \int_{-2}^{-1} \left[\sqrt{x+2} - (-\sqrt{x+2})\right] dx = 2\int_{-2}^{-1} \sqrt{x+2}\, dx = 2 \left.\frac{(x+2)^{3/2}}{3/2}\right]_{-2}^{-1} = \ldots$$

$$A(R_2) = \int_{-1}^{2} \left[\sqrt{x+2} - x\right] dx = \left[\frac{2}{3}(x+2)^{3/2} - \frac{x^2}{2}\right]_{-2}^{-1} = \ldots$$

$$\boxed{\text{Total area} = A(R_1) + A(R_2)}$$

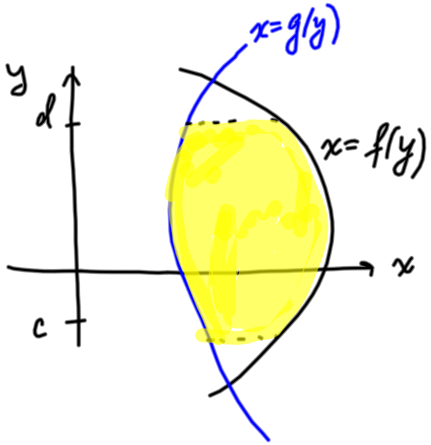
Integrate for y.



$$A(R) = \int_{-1}^{2} \left[y - (y^2 - 2)\right] dy$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3} + 2y\right]_{-1}^{2} = \ldots$$

If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous functions and $f(y) \geq g(y)$ for all y in $[c, d]$, then its area is



$$\boxed{A = \int_c^d [f(y) - g(y)]dy}$$