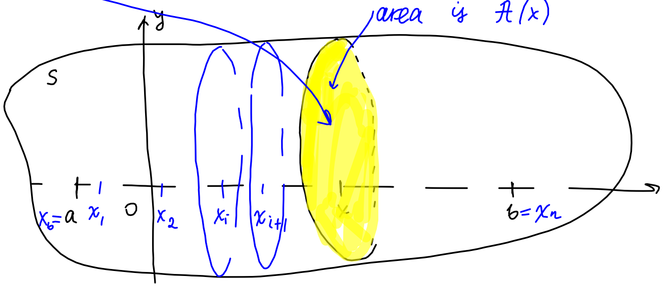
Section 7.2 **Volume**

We start with a simple type of solid called a **cylinder**. A cylinder is bounded by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments perpendicular to the base that join B_1 and B_2. If the area of the base is A and the height of the cylinder is h, then the volume of the cylinder is defined as $V = Ah$.

Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S. Suppose that the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x is $A(x)$, where $a \le x \le b$.

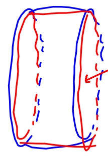
area is $A(x)$



$a \le x \le b$

interval $[x_i, x_{i+1}]$

$x_i^* \in [x_i, x_{i+1}]$



area of the base $= A(x_i^*)$

height $= \Delta x_i$

Volume $= A(x_i^*)\Delta x_i$

Let's consider a partition P of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$. The planes P_{x_i} will slice S into smaller "slabs". If we choose x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S between $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and height $\Delta x_i = x_i - x_{i-1}$.

The volume of this cylinder is $A(x_i^*)\Delta x_i$, so the approximation to volume of the ith slab is $V(S_i) \approx A(x_i^*)\Delta x_i$. Thus, the approximation to the volume of S is $\boxed{V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i.}$ This approximation appears to become better and better as $\|P\| \to 0$.

Definition of volume Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is an integrable function, then the **volume** of S is

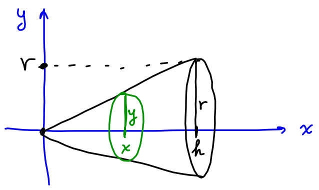
$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \boxed{\int_a^b A(x)dx = V}$$

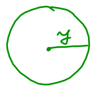
$A(x)$ is the area of a cross-sectional

IMPORTANT. $A(x)$ is the area of a moving cross-sectional obtained by slicing through x perpendicular to the x-axis.

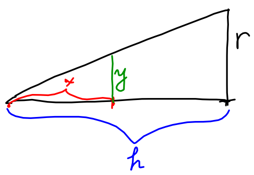
Example 1. Find the volume of a right circular cone with height h and base radius r.

Take $0 \le x \le h$





cross-sectional Area $= \pi y^2$



similar triangles:

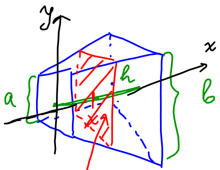
$$\frac{y}{x} = \frac{r}{h} \quad \text{- solve for } y$$

$$y = \frac{r}{h} x$$

$$\text{Area} = \pi \left(\frac{r}{h} x\right)^2 = \frac{\pi r^2}{h^2} x^2$$

$$\text{Volume} = \int_0^h \frac{\pi r^2}{h^2} x^2 \, dx = \left[\frac{\pi r^2}{h^2} \frac{x^3}{3}\right]_0^h = \boxed{\frac{\pi r^2 h}{3}}$$

Example 2. Find the volume of a frustum of a pyramid with square base of side b, square top of side a, and height h.

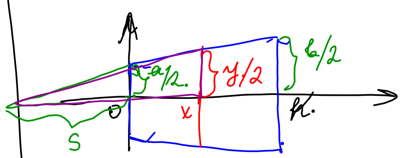


find its area

$A = y^2$

$0 \le x \le h$

Projection onto $(x\text{-}y)$-plane



similar triangles:

$$\frac{\frac{a}{2}}{s} = \frac{\frac{y}{2}}{s+x} = \frac{\frac{b}{2}}{s+h} \quad (2)$$

$$\frac{a}{s} = \frac{y}{s+x} = \frac{b}{s+h}$$

$$\frac{a}{s} = \frac{b}{s+h}$$

solve for s:

$$a(s+h) = bs$$

$$s(b-a) = ah$$

$$s = \frac{ah}{b-a}$$

$$\frac{a}{s} = \frac{y}{s+x} \qquad \text{solve for } y$$

$$y = \frac{a(s+x)}{s} = a + \frac{ax}{s}$$

$$= a + \frac{ax}{\frac{ah}{b-a}}$$

$$\boxed{y = a + \frac{x}{h}(b-a)}$$

$$A = y^2 = \left[a + \frac{x}{h}(b-a)\right]^2$$

$$= a^2 + 2\frac{ax}{h}(b-a) + \frac{x^2}{h^2}(b-a)^2$$

$$V = \int_0^h A\,dx = \int_0^h \left[a^2 + \frac{2a}{h}(b-a)x + \frac{(b-a)^2}{h^2}x^2\right]dx$$

$$= \left[a^2 x + \frac{\cancel{2}a}{h}(b-a)\frac{x^2}{\cancel{2}} + \frac{(b-a)^2}{h^2}\frac{x^3}{3}\right]_0^h$$

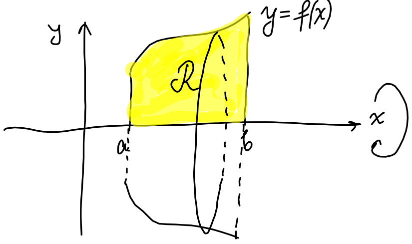
$$= a^2 h + \frac{a}{\cancel{h}}(b-a)h^{\cancel{2}} + \frac{(b-a)^2}{\cancel{h^2}}\frac{h^{\cancel{3}}}{3}$$

$$= a^2h + a(b-a)h + \underbrace{(b-a)^2}_{b^2-2ab+a^2}\frac{h}{3}$$

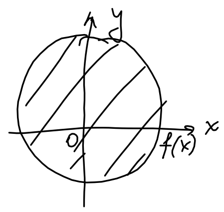
$$= \cancel{a^2h} + abh - \cancel{a^2h} + \frac{b^2}{3}h - \frac{2ab}{3}h + \frac{a^2}{3}h$$

$$= \boxed{\frac{h}{3}(b^2 + ab + a^2)}$$

Volume by disks. Let S be the solid obtained by revolving the plane region $\mathcal{R}$ bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the x-axis.



cross-sectional



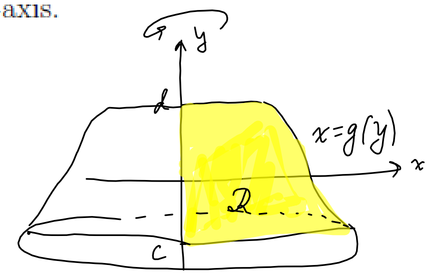
Radius $= f(x)$

$A = \pi r^2 = \pi [f(x)]^2$

A cross-section through x perpendicular to the x-axis is a circular disc with radius $|y| = |f(x)|$, the cross-sectional area is $A(x) = \pi y^2 = \pi[f(x)]^2$, thus, we have the following **formula for a volume of revolution:**

$$\boxed{V_X = \pi \int_a^b [f(x)]^2 dx}$$

The region bounded by the curves $x = g(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the y-axis.

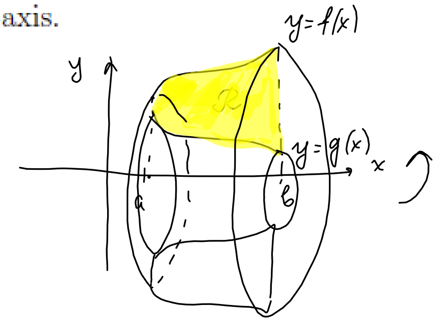


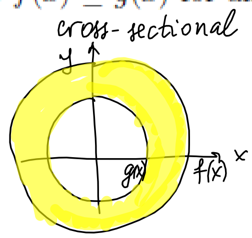
$$\boxed{V = \pi \int_c^d [g(y)]^2 dy}$$

Then the corresponding volume of revolution is

$$\boxed{V_Y = \pi \int_c^d [g(y)]^2 dy}$$

Volume by washers. Let S be the solid generated when the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ (where $f(x) \geq g(x)$ for all x in $[a, b]$) is rotated about the x-axis.





outer radius = $f(x)$
inner radius = $g(x)$

$A = \pi\left[(f(x))^2 - (g(x))^2\right]$

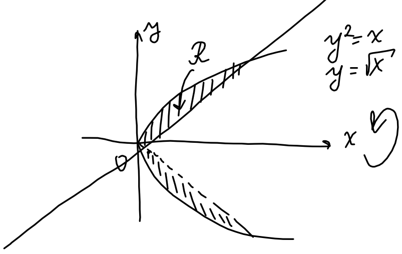
Then the volume of S is

$$\boxed{V_X = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\} dx}$$

Example 3.

1. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$, $x = 2y$ about the x-axis.

$2y = x$, $y = \frac{x}{2}$



$$V_x = \pi \int_0^4 [(\text{outer radius})^2 - (\text{inner radius})^2]\,dx$$

Points of intersection

$y^2 = 2y.$

$y(y-2) = 0$

$y_1 = 0, \quad y_2 = 2.$

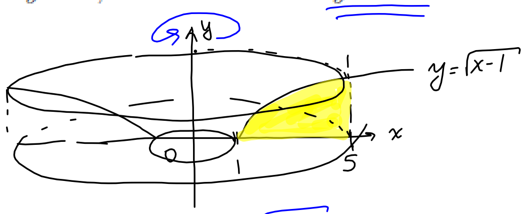
$x_1 = 0, \quad x_2 = 4$

outer radius $= \sqrt{x}$

inner radius $= \frac{x}{2}$

$$V_x = \pi \int_0^4 \left[(\sqrt{x})^2 - \left(\frac{x}{2}\right)^2 \right] dx = \pi \int_0^4 \left[x - \frac{x^2}{4} \right] dx = \dots$$

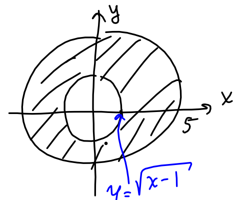
2. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about the y-axis.



$0 \le y \le \sqrt{5-1}$

$0 \le y \le 2$

cross sectional



$y = \sqrt{x-1}$

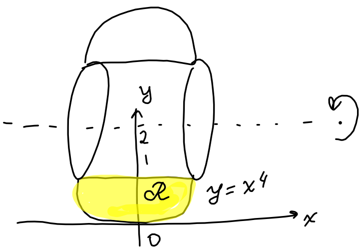
$y^2 = x-1$

$x = y^2+1$ inner radius.

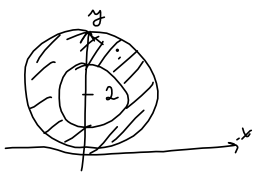
$$V_y = \pi \int_0^2 \left[25 - (y^2+1)^2\right] dy$$

$$= \pi \int_0^2 \left[25 - y^4 - 2y^2 - 1\right] dy = \ldots$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 1$ about the line $y = 2$.



cross-sectional:



outer radius $= 2 - x^4$

inner radius $= 1$

$y=2$ is parallel to x-axis.

$$V_{(y=2)} = \pi \int_{-1}^{1} \left[(2-x^4)^2 - 1 \right] dx = \dots$$