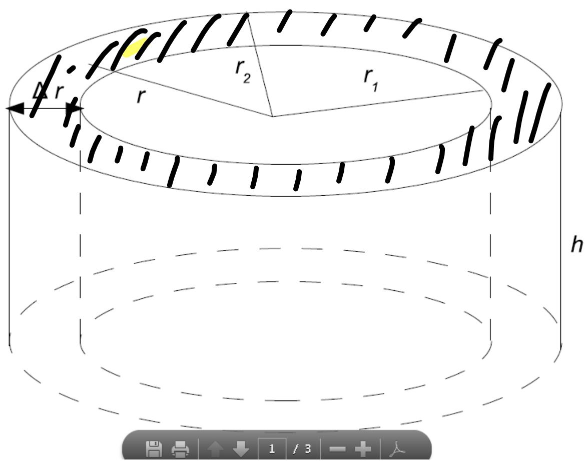
Section 7.3 **Volumes by cylindrical shells**

Lets find the volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h (see Fig.1).



$$V_{shell} = \pi r_2^2 h - \pi r_1^2 h$$
$$= \pi h (r_2^2 - r_1^2)$$
$$= \pi h \underbrace{(r_2 - r_1)}_{\Delta r} \underbrace{(r_2 + r_1)}_{2r}$$

r is the average radius.

$$= \boxed{2\pi \Delta r \, r \, h}$$

V can be calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1 = \pi h(r_2^2 - r_1^2) = 2\pi h \frac{r_2 + r_1}{2}(r_2 - r_1)$$

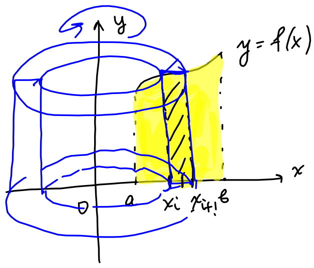
Let $\Delta r = r_2 - r_1$, $r = (r_2 + r_1)/2$, then the volume of a cylindrical shell is

$$\boxed{V = 2\pi r h \Delta r}$$

$V =$[circumference][height][thickness]

$\boxed{V = 2\pi\text{[average radius][height][thickness]}}$

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$.



1) Partition $[a,b]$ into n subintervals.
$a = x_0 < x_1 < \ldots < x_n = b$

2) Take an interval $[x_i, x_{i+1}]$, $i = 0,1,2,\ldots,n-1$
do the lines $x = x_i$, $x = x_i^*$
the thickness of the strip is $\Delta x_i = x_{i+1} - x_i$

3) approximate the strip by a rectangle with sides Δx_i and $f(x_i^*)$, where x_i^* $[x_i, x_{i+1}]$.

4) Rotate the rectangle about the y-axis. We'll get a cylindrical shell with the volume

$$V_i = 2\pi\, x_i^* f(x_i^*)\, \Delta x_i$$

5) $V \approx \sum_{i=1}^{n} 2\pi\, x_i^* f(x_i^*)\, \Delta x_i$

lim as $\|P\| \to 0$

$$\boxed{V_y = 2\pi \int_a^b x\, f(x)\, dx}$$

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < \ldots < x_n = b$ and let x_i^* be the midpoint of $[x_{i-1}, x_i]$, that is $x_i^* = (x_{i-1} + x_i)/2$. If the rectangle with base $[x_{i-1}, x_i]$

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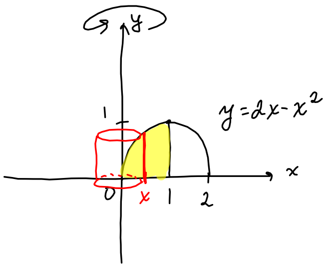
and height $f(x_i^*)$ is rotated about the y-axis, then the result is a cylindrical shell with average raduis x_i^*, height $f(x_i^*)$, and thikness $\Delta x_i = x_i - x_{i-1}$, so its volume is $V_i = 2\pi x_i^* f(x_i^*)\Delta x_i$.

The approximation to the volume V of S is $V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

Thus, the volume of S is

$$V_Y = \lim_{\|P\|\to 0} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i = 2\pi \int_a^b x f(x)dx$$

Example 1. Find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$, $y = 0$, $x = 0$, $x = 1$ about the y-axis.



$V_y = ?$

$0 \le x \le 1$

$$V_y = 2\pi \int_0^1 \underbrace{[\text{radius}]}_{x}\underbrace{[\text{height}]}_{2x-x^2}\,dx$$

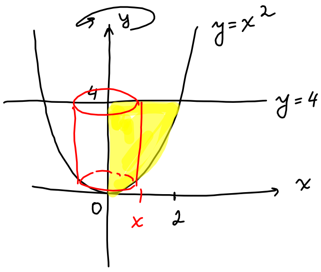
$$= 2\pi \int_0^1 x(2x - x^2)\,dx$$

$$= 2\pi \int_0^1 (2x^2 - x^3)\,dx = \dots$$

The volume of the solid generated by rotating about the y-axis the region between the curves $y = f(x)$ and $y = g(x)$ from a to b [$f(x) \geq g(x)$ and $0 \leq a < b$] is

$$\boxed{V_Y = 2\pi \int_a^b x[f(x) - g(x)]dx}$$

Example 2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 4$, $x = 0$ about the y-axis, $x > 0$.

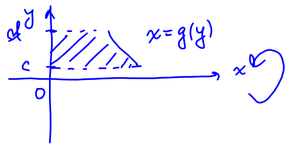


$$V_y = 2\pi \int_0^2 \underbrace{[\text{radius}]}_{x}\underbrace{[\text{height}]}_{4-x^2} dx$$

$$= 2\pi \int_0^2 x(4-x^2)\,dx$$

$$= 2\pi \int_0^2 (4x - x^3)\,dx = \dots$$

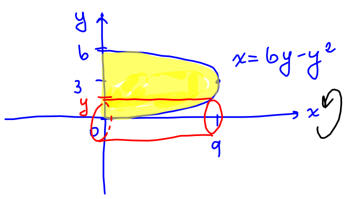
The method of cylindrical shells also allows us to compute volumes of revolution about the x-axis. If we interchange the roles of x and y in the formula for the volume, then the volume of the solid generated by rotating the region bounded by $x = g(y)$, $x = 0$, $y = c$, and $y = d$ about the x-axis, is



$$\boxed{V_X = 2\pi \int_c^d y g(y) dy}$$

Example 3. Find the volume of the solid obtained by rotating the region bounded by $y^2 - 6y + x = 0$, $x = 0$ about the x-axis.

$x = 6y - y^2$



$0 \le y \le 6$

$$V_x = 2\pi \int_0^6 \underbrace{[\text{radius}]}_{y}\underbrace{[\text{height}]}_{6y-y^2} dy$$

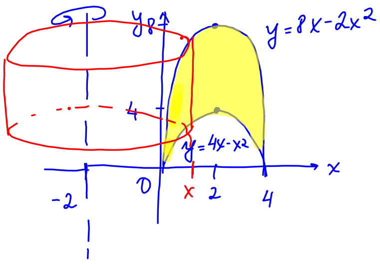
$$= 2\pi \int_0^6 y(6y - y^2) dy$$

$$= 2\pi \int_0^6 (6y^2 - y^3) dy = \dots$$

The volume of the solid generated by rotating the region bounded by $x = g_1(y)$, $x = g_2(y)$, $y = c$, and $y = d$, about the x-axis, assuming that $g_2(y) \geq g_1(y)$ for all $c \leq x \leq d$, is

$$\boxed{V_X = 2\pi \int_c^d y[g_2(y) - g_1(y)]dy}$$

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$, $y = 8x - 2x^2$ about $x = -2$.



$0 \leq x \leq 4$

$$V_{x=-2} = 2\pi \int_0^4 \underbrace{[\text{radius}]}_{x+2}\underbrace{[\text{height}]}_{\substack{8x-2x^2-(4x-x^2)\\ 4x-x^2}} dx$$

$$= 2\pi \int_0^4 (x+2)(4x-x^2)\,dx$$

$$= 2\pi \int_0^4 (4x^2 - x^3 + 8x - 2x^2)\,dx$$

$$= 2\pi \int_0^4 (2x^2 - x^3 + 8x)\,dx = \ldots$$