Section 7.4 **Work**

Mechanical work is the amount of energy transferred by a force.

If an object moves along a straight line with position function $s(t)$, then the force F on the object (in the same direction) is defined by Newton's Second Law of Motion

$$F = ma = m\frac{d^2s}{dt^2}$$

In case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves

$$W = Fd, \text{ work=force } \times \text{ distance}$$

Mechanical units in the U.S. customary and SI metric systems

Unit	**U.S. customary system**	**SI metric system**
distance	ft	m
mass	$slug$	kg
force	lb	$N = kg \cdot m/sec^2$
work	$ft - lb$	$J = N \cdot m$
g(Earth)	$32ft/sec^2$	$9.81m/sec^2$

What happens if the force is variable?

Problem The object moves along the x-axis in the positive direction from $x = a$ to $x = b$ and at each point x between a and b a force $f(x)$ acts on the object, where f is continuous function. Find the work done in moving the object from a to b.

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < \ldots < x_n = b$ and let $\Delta x_i = x_i - x_{i-1}$, and let x_i^* is in $[x_{i-1}, x_i]$. Then the force at x_i^* is $f(x_i^*)$. If $\|P\|$ is small, then Δx_i is small, and since f is continuous, the values of f do not change very much on $[x_{i-1}, x_i]$. In other words f is almost a constant on the interval and so work W_i that is done in moving the particle from x_{i-1} to x_i is $W_i \approx f(x_i^*)\Delta x_i$. We can approximate the total work by

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This approximation becomes better and better as $\|P\| \to 0$.

Therefore, we define the **work done in moving the object from a to b** as

$$W = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)dx$$

Example 1. When a particle is at a distance x meters from the origin, a force of $\cos(\pi x/3)$ N acts on it. How much work is done by moving the particle from $x = 1$ to $x = 2$.

Hooke's Law: The force required to maintain a spring stretched x units beyond its natural length is proportional to x

$$\boxed{f(x) = kx}$$

where k is a positive constant (the **spring constant**).

Example 2. Suppose that 2 J of work are needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch it from 35 cm to 40 cm?

2 J is needed to stretch $30_{cm} \to 42$ cm

30 cm is the natural length

$35_{cm} \to 40$ cm $-$?

$f(x) = kx$, k is unknown

$30 \leftrightarrow 0$

$42 \leftrightarrow 42 - 30 = 12(cm) = 0.12(m)$

$W = \int_0^{0.12} f(x)\,dx$

$$2 = \int_0^{0.12} kx\,dx = k\frac{x^2}{2}\Big|_0^{0.12}$$

$$2 = k\frac{0.0144}{2}$$

$$\boxed{k = \frac{4}{0.0144}} = \frac{2500}{9}$$

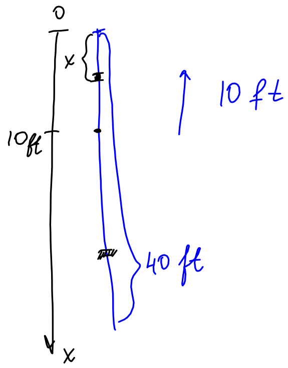
$35 \leftrightarrow 35 - 30 = 5(cm) = 0.05(m)$

$40 \leftrightarrow .1(m)$

$$W = \int_{0.05}^{0.1} \frac{2500}{9} x\,dx = \frac{2500}{9}\frac{x^2}{2}\Big|_{0.05}^{0.1}$$

$$= \boxed{\frac{2500}{9} \cdot \frac{1}{2}[0.01 - 0.0025]}$$

Example 3. A uniform cable hanging over the edge of a tall building is 40 ft long and weights 60 lb. How much work is required to pull 10 ft of the cable to the top?



1) $0 \le x \le 10$

piece of the cable Δx thick

its weight $= \frac{60}{40}\Delta x$

$= \frac{3}{2}\Delta x$

distance travelled $= x$

$\sum_{i=1}^{n} \frac{3}{2} x \Delta x \xrightarrow{\|P\| \to 0} \int_0^{10} \frac{3}{2} x\,dx$

Work done by pulling up first 10 ft of the cable

$W_1 = \int_0^{10} \frac{3}{2} x\,dx$

$= \frac{3}{2} \frac{x^2}{2} \Big|_0^{10}$

$= \boxed{75 \text{ (ft-lb)}}$

2) $10 < x \le 40$

piece of the cable Δx thick

weight $= \frac{3}{2}\Delta x$

distance travelled $= 10$

$W_2 = \int_{10}^{40} \frac{3}{2}(10)\,dx$

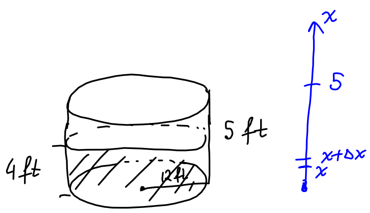
$= \frac{30}{2} x \Big|_{10}^{40}$

$= 15(30)$

$= \boxed{450 \text{ (ft-lb)}}$

3) total work $= \boxed{450 + 75}$

Example 4. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all the water out over the side?

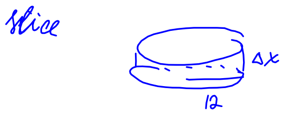


$0 \le x \le 4$

Slice of the water from x to $x+\Delta x$

distance travelled $= 5-x$

weight $= \underbrace{\pi(12)^2 \Delta x}_{\text{volume of the slice}}\ (62.5)$ ← density of water



$$W = \int_0^4 (5-x)(62.5)\pi\, 144\, dx$$

$= \ldots$