

Section 7.5 Average value of a function

Let us try to compute the average value of a function $y = f(x)$, $a \leq x \leq b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b-a)/n$ and choose points x_i^* in successive subintervals. Then the average of the numbers $f(x_1^*), f(x_2^*), \dots, f(x_n^*)$, is

$$\frac{f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)}{n}$$

Since $n = (b-a)/\Delta x$,

$$\frac{f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a} (f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x)$$

The limiting value as $n \rightarrow \infty$ is

$$\lim_{n \rightarrow \infty} \frac{1}{b-a} \sum_{i=1}^n f(x_i^*)\Delta x = \frac{1}{b-a} \int_a^b f(x)dx$$

We define the **average value of f** on the interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

Example 1. Find the average value of $f(x) = \sin^2 x \cos x$ on $[\pi/4, \pi/2]$.

$$\begin{aligned} f_{ave} &= \frac{1}{\frac{\pi}{2} - \frac{\pi}{4}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x \cos x dx \\ &= \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x \cos x dx \quad \left\{ \begin{array}{l} u = \sin x \\ du = \cos x dx \\ \frac{\pi}{4} \rightarrow \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ \frac{\pi}{2} \rightarrow \sin \frac{\pi}{2} = 1 \end{array} \right. \\ &= \frac{4}{\pi} \int_{\frac{\sqrt{2}}{2}}^1 u^2 du = \frac{4}{\pi} \left. \frac{u^3}{3} \right|_{\frac{\sqrt{2}}{2}}^1 = \frac{4}{3\pi} \left[1 - \left(\frac{\sqrt{2}}{2} \right)^3 \right] \\ &= \boxed{\frac{4}{3\pi} \left[1 - \frac{\sqrt{2}}{4} \right]} \end{aligned}$$

Example 2. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

Mean value theorem for integrals If f continuous on $[a, b]$, then there exist a number c in $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b - a)$$

The geometric interpretation of this theorem for *positive* functions $f(x)$, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as a region under the graph of f from a to b .

Example 3. Find the average value of the function $f(x) = 4 - x^2$ on the interval $[0, 2]$. Find c ($0 \leq c \leq 2$) such that $f_{ave} = f(c)$.