

1. Evaluate the integral

$$(a) \int \frac{x^2}{\sqrt{5-x^2}} dx = \left| \begin{array}{l} x = 5 \sin t \\ dx = 5 \cos t dt \\ \sqrt{5-x^2} = 5 \cos t \end{array} \right| = \int \frac{25 \sin^2 t}{5 \cos t} 5 \cos t dt$$

$$= \int 25 \sin^2 t dt = \frac{25}{2} \int (1 - \cos 2t) dt = \frac{25}{2} \left( t - \frac{1}{2} \sin 2t \right) + C$$

$$\frac{1}{2} \sin 2t = \sin t \cos t$$

$$\text{since } \sin t = \frac{x}{5}, \text{ then } \cos t = \sqrt{1-\sin^2 t} = \frac{1}{5} \sqrt{5-x^2}$$

$$t = \sin^{-1}\left(\frac{x}{5}\right), \text{ then}$$

$$\int \frac{x^2}{\sqrt{5-x^2}} dx = \frac{25}{2} \left( \sin^{-1}\left(\frac{x}{5}\right) - \frac{x}{25} \sqrt{5-x^2} \right) + C$$

$$(b) \int \frac{x^3}{\sqrt{x^2+4}} dx = \left| \begin{array}{l} x^2+4=u \\ du=2x dx \\ x^2=u-4 \end{array} \right| = \frac{1}{2} \int \frac{u-4}{\sqrt{u}} du$$

$$= \frac{1}{2} \int (u^{1/2} - 4u^{-1/2}) du = \frac{1}{2} \left( \frac{u^{3/2}}{3/2} - 4 \frac{u^{1/2}}{1/2} \right) + C$$

$$= \frac{1}{2} \left( \frac{2}{3} (x^2+4)^{3/2} - 8 \sqrt{x^2+4} \right) + C$$

OR

$$\left| \begin{array}{l} x = 2 \tan t \\ dx = 2 \sec^2 t dt \\ \sqrt{x^2+4} = 2 \sec t \end{array} \right| = \int \frac{8 \tan^3 t}{2 \sec t} 2 \sec^2 t dt = 8 \int \tan^3 t \sec t dt$$

$$\left| \begin{array}{l} u = \sec t \\ du = \sec t \tan t dt \\ \tan^2 t = \sec^2 t - 1 = u^2 - 1 \end{array} \right| \quad \left| \begin{array}{l} = 8 \int (u^2 - 1) du = 8 \left( \frac{u^3}{3} + u \right) + C \\ = 8 \left( \frac{\sec^3 t}{3} + \sec t \right) + C \\ = 8 \left( \frac{1}{3} (4+x^2)^{3/2} + \sqrt{4+x^2} \right) + C \end{array} \right.$$

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$$\tan t = \frac{x}{2}$$

$$\sec t = \sqrt{1+\tan^2 t} = \frac{1}{2} \sqrt{4+x^2}$$

$$(c) \int \frac{dx}{\sqrt{x^2 + 4x - 5}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2 - 9}}$$

$$= \int \frac{3 \sec t \tan t dt}{3 \tan t} = \int \sec t dt = \ln |\sec t + \tan t| + C$$

$$\sec t = \frac{x+2}{3}$$

$$\tan t = \sqrt{\sec^2 t - 1} = \sqrt{\frac{(x+2)^2}{9} - 1} = \frac{1}{3} \sqrt{(x+2)^2 - 9}$$

$$(d) \int \frac{dx}{x^2(x^2 + 1)}$$

Partial fraction decomposition:

$$\begin{aligned} \frac{1}{x^2(x^2 + 1)} &= \frac{A}{x^2} + \frac{B}{x^2 + 1} + \frac{Cx + D}{x^2 + 1} = \frac{1}{x^2} - \frac{1}{x^2 + 1} \\ &= \frac{Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2}{x^2 + 1} \\ &= \frac{x^3(A + C) + x^2(B + D) + x(A) + B}{x^2 + 1} \end{aligned}$$

$$x^3: A + C = 0$$

$$x^2: B + D = 0$$

$$x^1: A = 0$$

$$x^0: B = 1$$

$$\boxed{A=0} \\ \boxed{C=-A=0}$$

$$\boxed{B=1} \\ \boxed{D=-B=-1}$$

$$\int \frac{dx}{x^2(x^2 + 1)} = \int \left( \frac{1}{x^2} - \frac{1}{x^2 + 1} \right) dx = -\frac{1}{x} - \tan^{-1}(x) + C$$

$$(e) \int_0^\infty \frac{dx}{(x+2)(x+3)} = \lim_{t \rightarrow \infty} \left[ \int_0^t \frac{dx}{(x+2)(x+3)} \right] - \text{convergent.}$$

Partial fraction decomposition:

$$\begin{aligned} \frac{1}{(x+2)(x+3)} &= \frac{A}{x+2} + \frac{B}{x+3} = \frac{1}{x+2} - \frac{1}{x+3} \\ &= \frac{A(x+3) + B(x+2)}{(x+2)(x+3)} \end{aligned}$$

$$x = -3: 1 = -B \rightarrow B = -1$$

$$x = -2: 1 = A \rightarrow A = 1$$

$$\begin{aligned} &= \lim_{t \rightarrow \infty} \left[ \int_0^t \left( \frac{1}{x+2} - \frac{1}{x+3} \right) dx \right] = \lim_{t \rightarrow \infty} \left( \ln|x+2| - \ln|x+3| \right) \Big|_0^t \\ &= \lim_{t \rightarrow \infty} (\ln|t+2| - \ln|t+3| - \ln 2 + \ln 3) = \lim_{t \rightarrow \infty} \frac{\ln \frac{t+2}{t+3}^0}{\ln 3 - \ln 2} = \boxed{\ln 2 + \ln 3} \end{aligned}$$

$$(f) \int_{-\infty}^1 \frac{dx}{(2x-3)^2} - \text{convergent}$$

$$= \lim_{s \rightarrow -\infty} \left[ \int_s^1 \frac{dx}{(2x-3)^2} \right]$$

$$= \lim_{s \rightarrow -\infty} \left( -\frac{1}{2} \frac{1}{2x-3} \right) \Big|_s^1$$

$$= \lim_{s \rightarrow -\infty} \left( -\frac{1}{2} \left( (-1) - \frac{1}{2s-3}^0 \right) \right)$$

$$= \boxed{\frac{1}{2}}$$

$$(g) \int_4^5 \frac{dx}{(5-x)^{2/5}} - \text{convergent}$$

$$\begin{aligned}
 &= \lim_{t \rightarrow 5^-} \int_4^t \frac{dx}{(5-x)^{2/5}} \\
 &= \lim_{t \rightarrow 5^-} -\frac{(5-x)^{-2/5+1}}{-2/5+1} \Big|_4^t = \lim_{t \rightarrow 5^-} \frac{5(5-t)^{3/5}}{3} \Big|_4^t \\
 &= -\lim_{t \rightarrow 5^-} \frac{5}{3} \left( \cancel{(5-t)^{3/5}} - 1 \right) = \boxed{+\frac{5}{3}}
 \end{aligned}$$

2. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x - 1}{(x^2 - 1)(x + 1)(x^2 + 1)^2}.$$

Do not determine the numerical values for the coefficients.

$$\begin{aligned}
 (x^2 - 1)(x + 1)(x^2 + 1)^2 &= (x-1)(x+1)(x+1)(x^2+1)^2 \\
 &= (x-1)(x+1)^2(x^2+1)^2
 \end{aligned}$$

$$\begin{aligned}
 \frac{x^3 + x - 1}{(x^2 - 1)(x + 1)(x^2 + 1)^2} &= \frac{x^3 + x - 1}{(x-1)(x+1)^2(x^2+1)^2} \\
 &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}
 \end{aligned}$$

3. Find the length of the curve  $x(t) = 3t - t^3$ ,  $y(t) = 3t^2$ ,  $0 \leq t \leq 2$ .

$$L = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

$$x'(t) = 3 - 3t^2$$

$$y'(t) = 6t$$

$$\begin{aligned}[x'(t)]^2 + [y'(t)]^2 &= (3 - 3t^2)^2 + 36t^2 \\&= 9 - 18t^2 + 9t^4 + 36t^2 \\&= 9 + 18t^2 + 9t^4 \\&= (3 + 3t^2)^2\end{aligned}$$

$$L = \int_0^2 (3 + 3t^2) dt = \left( 3t + \frac{3t^3}{3} \right) \Big|_0^2$$

$$= (6 + 8)$$

$$= \boxed{14}$$

4. Find the area of the surface obtained by rotating the curve  $y = x^3$ ,  $0 \leq x \leq 2$  about the  $x$ -axis.

$$S_x = 2\pi \int_0^2 y(x) \sqrt{1 + [y'(x)]^2} dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2} dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4} dx$$

$$\left| \begin{array}{l} u = 1 + 9x^4 \\ du = 36x^3 dx \\ x=0 \rightarrow u=1 \\ x=2 \rightarrow u = 1 + 9(16) = 145 \end{array} \right|$$

$$= \frac{2\pi}{36} \int_1^{145} \sqrt{u} du$$

$$= \cancel{\frac{2\pi}{36}} \left[ \frac{\pi}{18} u^{3/2} \right]_1^{145}$$

$$= \frac{\pi}{27} ((145)^{3/2} - 1)$$

5. Find the area of the surface obtained by rotating the curve  $x = \sqrt{2y - y^2}$ ,  $0 \leq y \leq 1$  about the  $y$ -axis.

$$S_y = 2\pi \int_0^1 x(y) \sqrt{1 + (x'(y))^2} dy$$

$$x'(y) = \frac{1}{2\sqrt{2y-y^2}} (2-2y) = \frac{1-y}{\sqrt{2y-y^2}}$$

$$1 + [x'(y)]^2 = 1 + \frac{(1-y)^2}{2y-y^2} = \frac{2y-y^2+1-2y+y^2}{2y-y^2} \\ = \frac{1}{2y-y^2}$$

$$S_y = 2\pi \int_0^1 \sqrt{2y-y^2} \cdot \frac{1}{\sqrt{2y-y^2}} dy$$

$$\boxed{2\pi}$$

6. Find  $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\ln n}$  *l'Hospital's Rule*

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{2\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2} = \infty$$

7. Find the sum of the series

$$\begin{aligned} \text{(a)} \sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^{3n-1}} &= \sum_{n=1}^{\infty} \frac{2 \cdot 2^{2n}}{\frac{1}{3} 3^{3n-1}} \\ &= \sum_{n=1}^{\infty} 6 \cdot \frac{(2^2)^n}{(3^3)^n} \\ &= 6 \sum_{n=1}^{\infty} \frac{4^n}{27^n} \\ &= 6 \sum_{n=1}^{\infty} \left(\frac{4}{27}\right)^n \\ &= 6 \sum_{n=1}^{\infty} \frac{4}{27} \left(\frac{4}{27}\right)^{n-1} \\ &= \cancel{6} \cancel{\sum_{n=1}^{\infty} \frac{4}{27} \left(\frac{4}{27}\right)^{n-1}} \\ &= \frac{24}{27} \cdot \frac{1}{1 - \frac{4}{27}} \\ &= \frac{8}{9} \cdot \frac{27}{23} \\ &= \boxed{\frac{24}{23}} \end{aligned}$$

$$(b) \sum_{n=3}^{\infty} \frac{1}{n^2 - 4}$$

$$\frac{1}{n^2 - 4} = \frac{1}{(n-2)(n+2)} = \frac{A}{n-2} + \frac{B}{n+2} = \frac{1}{4} \left( \frac{1}{n-2} + \frac{1}{n+2} \right)$$

$$= \frac{A(n+2) + B(n-2)}{(n-2)(n+2)}$$

$$n=-2: 1 = -4B \rightarrow B = -\frac{1}{4}$$

$$n=2: 1 = 4A \rightarrow A = \frac{1}{4}$$

Partial sum:

$$S_3 = \frac{1}{4} \left( \frac{1}{1} + \frac{1}{5} \right)$$

$$S_4 = \frac{1}{4} \left( \frac{1}{1} + \frac{1}{5} \right) + \frac{1}{4} \left( \frac{1}{2} + \frac{1}{6} \right)$$

$$S_5 = \frac{1}{4} \left( \frac{1}{1} + \frac{1}{5} \right) + \frac{1}{4} \left( \frac{1}{2} + \frac{1}{6} \right) + \frac{1}{4} \left( \frac{1}{3} + \frac{1}{7} \right)$$

$$S_6 = \frac{1}{4} \left( \frac{1}{1} - \frac{1}{5} \right) + \frac{1}{4} \left( \frac{1}{2} - \frac{1}{6} \right) + \frac{1}{4} \left( \frac{1}{3} - \frac{1}{7} \right) + \frac{1}{4} \left( \frac{1}{4} - \frac{1}{8} \right)$$

$$S_7 = \frac{1}{4} \left( \frac{1}{1} - \frac{1}{5} \right) + \frac{1}{4} \left( \frac{1}{2} - \frac{1}{6} \right) + \frac{1}{4} \left( \frac{1}{3} - \frac{1}{7} \right) + \frac{1}{4} \left( \frac{1}{4} - \frac{1}{8} \right) + \frac{1}{4} \left( \frac{1}{5} - \frac{1}{9} \right)$$

$$\dots$$

$$S_n = \frac{1}{4} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) - \frac{1}{n+1} - \frac{1}{n+2}$$

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{4} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) \approx$$

$$= \boxed{\frac{25}{48}}$$

8. Determine whether the series is convergent or divergent.

(a)  $\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7} + 1}$  - diverges by Divergence Test.

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^{5/7} + 1} = \lim_{n \rightarrow \infty} \frac{n^2}{n^{5/7}(1 + \frac{1}{n^{5/7}})} = \lim_{n \rightarrow \infty} n^{9/7} = \infty$$

$$(b) \sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$$

$$D \leq \cos^2 n \leq 1$$

$$0 \leq \frac{\cos^2 n}{3^n} \leq \frac{1}{3^n}$$

~~compare~~ with

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$  converges, then  $\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$  converges  
 since  $\sum_{n=1}^{\infty} \frac{1}{3^n}$  converges, then by the Comparison Test I

(c)  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  — converges by The Integral Test  
 compare with  $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$

$$f(x) = \frac{1}{x(\ln x)^2} \text{ on } [2, \infty)$$

$$\text{D) } x > 2 \quad \ln x > 0 \quad \text{on } [2, \infty) \\ f(x) > 0 \quad \text{on } [2, \infty)$$

2)  $f(x)$  has discontinuity<sup>only</sup> at  $x=1$   
 &  $f(x)$  is continuous on  $[2, \infty)$ .

$$3) \text{ If } (x(\ln x)^2)' = (\ln x)^2 + x \cdot 2 \ln x \cdot \frac{1}{x} = (\ln x)^2 + 2 \ln x > 0 \text{ on } [2, \infty)$$

since  $x(\ln x)^2$  is increasing on  $[2, \infty)$ , then  $\frac{1}{x(\ln x)^2}$  is decreasing on  $[2, \infty)$ .  $|u = \ln x| \quad |e^{\int \frac{1}{u^2} du}|$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx$$

is decreasing on  $[2, \infty)$ .

$$= \lim_{t \rightarrow \infty} \left( -\frac{1}{t} \right) \Big|_{\ln 2}^{\ln t} = \lim_{t \rightarrow \infty} \left( -\frac{1}{\ln t_1} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$$

converges.

9. Approximate the sum of the series  $\sum_{n=1}^{\infty} ne^{-n^2}$  by using the sum of first 4 terms. Estimate the error involved in this approximation.

$$\sum_{n=1}^{\infty} ne^{-n^2} \approx S_4$$

$$S_4 = 1e^{-1} + 2e^{-4} + 3e^{-9} + 4e^{-16}$$

$$\int_5^{\infty} xe^{-x^2} dx \leq R_4 \leq \int_4^{\infty} xe^{-x^2} dx$$

$$\int_n^{\infty} xe^{-x^2} dx = \lim_{t \rightarrow \infty} \int_n^t xe^{-x^2} dx = \left| \begin{array}{l} u = x^2 \\ du = -2x dx \end{array} \right|$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} \int_{-t^2}^0 e^u du$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} (e^{-t^2} - e^{-n^2})$$

$$= \frac{1}{2} e^{-n^2}$$

Thus,

$$\boxed{\frac{1}{2} e^{-25} \leq R_4 \leq \frac{1}{2} e^{-16}}$$