

Chapter 10. **Infinite sequences and series**  
Section 10.4 **Other Convergence Tests**

An **alternating** series is a series of the form

$$b_1 - b_2 + b_3 - b_4 + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} b_n,$$

where  $b_n > 0$  for all  $n$ .

**The Alternating Series Test** If the series  $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$  satisfies

(a)  $b_{n+1} \leq b_n$  for all  $n$       (b)  $\lim_{n \rightarrow \infty} b_n = 0$ ,

then the series is convergent.

**Example 1.** Test the series for convergence or divergence.

(a)  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2^n}$

(b)  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{6n-5}$

**Alternating series estimating theorem** If  $s = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$  is the sum of alternating series that satisfies the Alternating Series Test, then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

**Definition** A series  $\sum_{n=1}^{\infty} a_n$  is called **absolutely convergent** if the series  $\sum_{n=1}^{\infty} |a_n|$  is convergent.

**Theorem** If a series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then it is convergent.

**Example 2.** Determine whether the series is absolutely convergent.

(a)  $\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$

(b)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

**The Ratio Test** Given a series  $\sum_{n=1}^{\infty} a_n$ . Let

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

1. If  $L < 1$ , then the series is absolutely convergent
2. If  $L > 1$ , then the series is divergent
3. If  $L = 1$ , then the test is inconclusive.

**The Root Test** Given a series  $\sum_{n=1}^{\infty} a_n$ . Let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L.$$

1. If  $L < 1$ , then the series is absolutely convergent
2. If  $L > 1$ , then the series is divergent
3. If  $L = 1$ , then the test is inconclusive.

**Example 3.** Test the series for absolute convergence, convergence or divergence

(a)  $\sum_{n=1}^{\infty} \left( \frac{n}{3n+1} \right)^n$

(b)  $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$

$$(c) \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$(d) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{5^n}{n^2}$$

$$(e) \sum_{n=1}^{\infty} \frac{2^{n-1}}{n^n}$$