MATH 152 501–506 Spring 2011 Review for Test III

Section 10.4 **Other Convergence Tests**

An **alternating** series is a series of the form

$$b_1 - b_2 + b_3 - b_4 + ... = \sum_{n=1}^{\infty} (-1)^{n+1} b_n,$$

where $b_n > 0$ for all n.

The Alternating Series Test If the series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ satisfies

(a) $b_{n+1} \le b_n$ for all n (b) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

Example 1. Which of the following series is convergent

1. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

2. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{\sqrt{n-2}}$

3. $\sum_{n=0}^{\infty} (-1)^{n} \frac{2^{2n}}{3^{3n}}$

Alternating series estimating theorem If $s = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$ is the sum of alternating series that satisfies the Alternating Series Test, then

$$|R_n| = |s - s_n| \le b_{n+1}$$

Example 2. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+2}}$ by using the sum of the first three terms. Estimate the error involved in this approximation.

Definition. A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem. If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

The Ratio Test Given a series $\sum_{n=1}^{\infty} a_n$. Let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

1. If $L < 1$, then the series is absolutely convergent
2. If $L > 1$, then the series is divergent
3. If $L = 1$, then the test is inconclusive.

The Root Test Given a series $\sum_{n=1}^{\infty} a_n$. Let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

1. If $L < 1$, then the series is absolutely convergent
2. If $L > 1$, then the series is divergent
3. If $L = 1$, then the test is inconclusive.

Example 3. Determine whether the series $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ is absolutely convergent.

Section 10.5 **Power series**

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n + ...$$

Constants c_n are called the **coefficients** of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is called a **power series centered at** a or a **power series about** a.

A power series is convergent if $|x-a| < R$, where

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

or

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}$$

R is called the **radius of convergence**.

If $R = 0$, then the series converges only at one point $x = a$.

If $R = \infty$, then the series converges for all x.

If $R \neq 0$ and $R < \infty$, then the series converges if $a - R < x < a + R$. We also have to check the convergence at $x = a - R$ and $x = a + R$.

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series is convergent.

Example 4. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}.$$

Section 10.6 **Representation of a function as power series**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Theorem If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then

1. $\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)' = \sum_{n=0}^{\infty} (c_n(x-a)^n)' = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$

2. $\int \left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) dx = \sum_{n=0}^{\infty} \left(\int c_n(x-a)^n \, dx\right) = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$

Example 5. Find the power series representation for the function

$$f(x) = \tan^{-1}(x^2)$$

Section 10.7 **Taylor and Maclaurin series**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the **Taylor series of** f **at** a.

Example 6. Find the Taylor series for $f(x) = x^3 + 3x^2 + 2$ at $x = 2$.

If $a = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

The series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ is called the **Maclaurin series**.

Important Maclaurin Series

1. $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

2. $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

3. $\displaystyle\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

4. $\displaystyle\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

5. $\displaystyle(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + ... + \frac{m(m-1)...(m-n+1)}{n!}x^n + ...$

Example 7. Find the Maclaurin series for $f(x) = x\sin(x^3)$.

Example 8. Find the sum of the series

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n x^2}{n!}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

Example 9. Evaluate the indefinite integral as a power series $\int e^{x^2} dx$.

Section 10.9 **Applications of Taylor polynomials**

Suppose that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Let

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

T_n is called the n**th-degree Taylor polynomial of** f **at** a .

We can use a Taylor polynomial T_n to approximate f. But how good an approximation is? To answer this question we need to look at

$$|R_n| = |f(x) - T_n(x)|$$

1. If the series happen to be an alternating series, then

$$|R_n| \leq \frac{|f^{(n+1)(a)}|}{(n+1)!}|x-a|^{n+1}$$

2. In other cases we can use **Taylor's Inequality**, which says if $|f^{(n+1)}(x)| \leq M$, then

$$|R_n| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Example 10. Approximate $f(x) = \sin x$ by a Taylor polynomial of degree 5 at $\pi/4$. How accurate is this approximation if $0 \leq x \leq \pi/2$?

Chapter 11. **Three-dimensional analytic geometry and vectors.**

Section 11.1 **Three-dimensional coordinate system.**

The three dimensional coordinate system is determined by three coordinate axes x-axis, y-axis, and z-axes, that are perpendicular to each other. The direction of z-axis is determined by the **right-hand rule**: if your index finger points in the positive direction of the x-axis, middle finger points in the positive direction of the y-axis, then your thumb points in the positive direction of the z-axis.

The three coordinate axes determine the three **coordinate planes**. The xy-plane contains the x- and y-axes and its equation is $z = 0$, the xz-plane contains the x- and z-axes and its equation is $y = 0$, The yz-plane contains the y- and z-axes and its equation is $x = 0$.

In three dimensional space we represent the point P by the ordered triple (a, b, c) of real numbers, and we call a, b, and c the **coordinates** of P.

The distance formula in three dimensions. The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **midpoint** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$P_M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Equation of a sphere. An equation of a sphere with center (a, b, c) and radius R is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$$

Example 11. Find radius and center of sphere given by the equation

$$x^2 + y^2 + z^2 = 6x + 4y + 10z$$

Section 11.2 **Vectors and the dot product in three dimensions**

Definition. A **tree-dimensional vector** is an ordered triple $\vec{a} =< a_1, a_2, a_3 >$ of real numbers. The numbers a_1, a_2, and a_3 are called the **components** of $\vec{a}$.

A **representation** of the vector $\vec{a} =< a_1, a_2, a_3 >$ is a directed line segment $\vec{AB}$ from any point $A(x, y, z)$ to the point $B(x + a_1, y + a_2, z + a_3)$.

A particular representation of $\vec{a} =< a_1, a_2, a_3 >$ is the directed line segment $\vec{OP}$ from the origin to the point $P(a_1, a_2, a_3)$, and $\vec{a} =< a_1, a_2, a_3 >$ is called the **position vector** of the point $P(a_1, a_2, a_3)$.

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then $\boxed{\vec{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 >}$.
The **magnitude (length)** $|\vec{a}|$ of $\vec{a}$ is the length of any its representation.
The length of $\vec{a}$ is $\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$

The only vector with length 0 is the **zero vector** $\vec{0} =< 0, 0, 0 >$. This vector is the only vector with no specific direction.

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then

$$\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$$

$$c\vec{a} =< ca_1, ca_2, ca_3 >, \quad \text{where } c \text{ is a scalar}$$

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >$$

Let $\vec{\imath} =< 1, 0, 0 >$ and $\vec{\jmath} =< 0, 1, 0 >$, $\vec{k} =< 0, 0, 1 >$, $|\vec{\imath}| = |\vec{\jmath}| = |\vec{k}| = 1$.

$$\vec{a} =< a_1, a_2, a_3 >= a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$$

A **unit vector** is a vector whose length is 1.

A vector

$$\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|} \right\rangle$$

is a unit vector that has the same direction as $\vec{a} =< a_1, a_2, a_3 >$.

Definition. The **dot** or **scalar product** of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$. If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a} \cdot \vec{b} = 0$.

$\vec{a} \cdot \vec{b} > 0$ if and only if $0 < \theta < \pi/2$

$\vec{a} \cdot \vec{b} < 0$ if and only if $\pi/2 < \theta < \pi$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\pi/2$.

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Example 12. Find the angle between the vectors $\vec{a} = \vec{\imath} + \vec{\jmath} + 2\vec{k}$ and $\vec{b} = 2\vec{\jmath} - 3\vec{k}$.

The **direction angles** of a nonzero vector $\vec{a}$ are the angles α, β, and γ in the interval $[0, \pi]$ that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector $\vec{a}$.

$$\cos\alpha = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|}, \quad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

Example 13. Find the directional cosines for the vector $\vec{a} = -2\vec{\imath} + 3\vec{\jmath} + \vec{k}$.

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} < a_1, a_2, a_3 >$$

Example 14. Find the scalar and the vector projections of the vector $< 2, -3, 1 >$ onto the vector $< 1, 6, -2 >$.