Chapter 10. **Infinite sequences and series**
Section 10.3 **The Integral and Comparison Tests; Estimating Sums**

The Integral Test Suppose f is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

n does not have to be 1, it could be a different number. Function $f(x)$ has to be ultimately decreasing function, that is, decreasing for $x > N$.

Example 1. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

Example 2. Find the values of p for which the series is convergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^p}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

The Comparison Test I Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series, such that $0 < a_n \leq b_n$ for all n.

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent

(b) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

The Comparison Test II Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0.$$

Then either both series converge or both diverge.

Example 3. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2 - 4}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$

(c) $\sum_{n=1}^{\infty} \left(\frac{2}{n\sqrt{n}} + \frac{3}{n^3} \right)$

(d) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$

Estimating the sum of a series

Suppose we've been able to show that a series $\sum_{n=1}^{\infty} a_n$ converges by Integral Test. We want to find an approximation to the sum S of the series. We can approximate S by partial sums S_n. How good is such an approximation?

We need to estimate the size of the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + ...$$

R_n is the error made when the partial sum S_n is used to approximate S.

$$R_n = a_{n+1} + a_{n+2} + ... \leq \int_{n}^{\infty} f(x)dx$$

here $f(n) = a_n$. Similarly,

$$R_n = a_{n+1} + a_{n+2} + ... \geq \int_{n+1}^{\infty} f(x)dx$$

Remainder estimate for the integral test If $\sum_{n=1}^{\infty} a_n$ converges by the Integral Test, $\sum_{n=1}^{\infty} a_n = s$, $\sum_{k=1}^{n} a_k = s_n$, and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$

or

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_{n}^{\infty} f(x)dx$$

Example 4. (a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ by using the sum of first 5 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 10^{-5}?

Example 5. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to three decimal places.