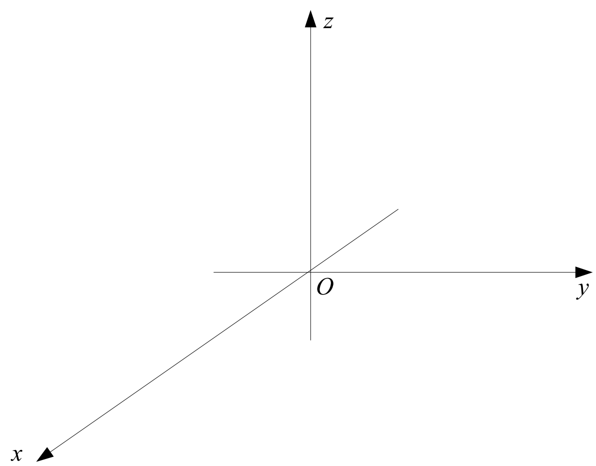
Chapter 11. **Three-dimensional analytic geometry and vectors**

Section 11.1 **Three-dimensional coordinate system**

To locate a point in space three numbers are required. We represent any point in space by an ordered triple (a, b, c).

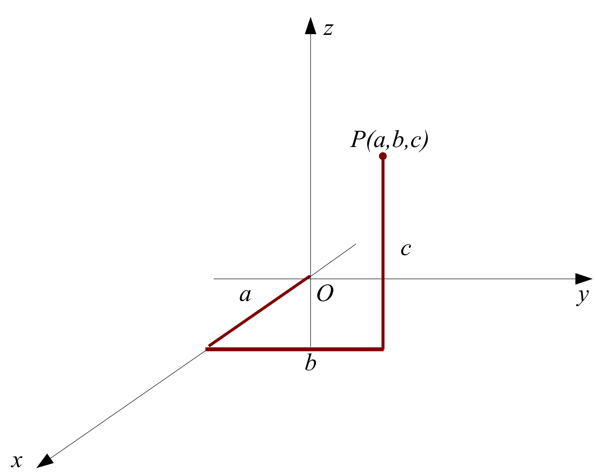
In order to represent points in space, we first choose a fixed point O (the origin) and tree directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x and y-axes as being horizontal and z-axis as being vertical.

The direction of z-axis is determined by the **right-hand rule**: if your index finger points in the positive direction of the x-axis, middle finger points in the positive direction of the y-axis, then your thumb points in the positive direction of the z-axis.



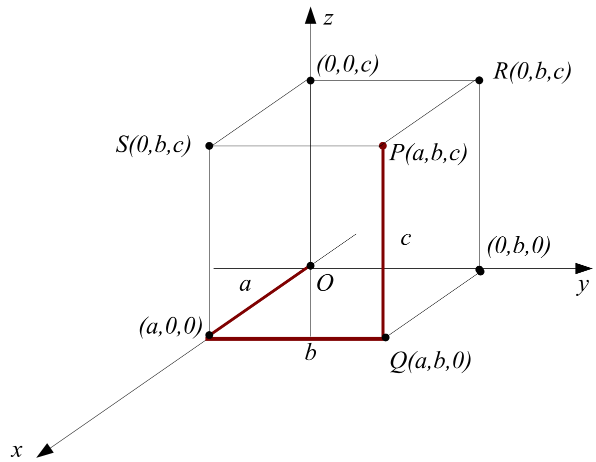
The three coordinate axes determine the three **coordinate planes**. The xy-plane contains the x- and y-axes and its equation is $z = 0$, the xz-plane contains the x- and z-axes and its equation is $y = 0$, The yz-plane contains the y- and z-axes and its equation is $x = 0$. These three coordinate planes divide space into eight parts called **octants**. The **first octant** is determined by positive axes.

Take a point P in space, let a be directed distance from yz-plane to P, b be directed distance from xz-plane to P, and c be directed distance from xy-plane to P.



We represent the point P by the ordered triple (a, b, c) of real numbers, and we call a, b, and c the **coordinates** of P.

The point $P(a, b, c)$ determine a rectangular box.



If we drop a perpendicular from P to the xy-plane, we get a point $Q(a, b, 0)$ called the **projection** of P on the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P on the yz-plane and xz-plane, respectively.

Example 1. Draw a rectangular box that has $P(1, 1, 2)$ and $Q(3, 4, 5)$ as opposite vertices and has its faces parallel to the coordinate planes. Then find the coordinates of the other six vertices of the box.

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 = \{(x, y, z)|x, y, z \in \mathbb{R}\}$ is the set of all ordered triplets of real numbers. We have given a one-to-one correspondence between points P in space and ordered triplets (a, b, c) in $\mathbb{R}^3$. It is called a **tree-dimensional rectangular coordinate system.**

Example 2. What surfaces in $\mathbb{R}^3$ represented by the following equations?

1. $x = 9$

2. $y = -1$

3. $z = 4$

4. $x + y = 1$

5. $z = x$

6. $x^2 + z^2 = 9$

7. $y = x^2$

The distance formula in three dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 3. Find the length of the sides of the triangle ABC, where $A(-2, 6, 1)$, $B(5, 4, -3)$, and $C(2, -6, 4)$.

Example 4. Determine whether the points $P(1, 2, 3)$, $Q(0, 3, 7)$, and $R(3, 5, 11)$ are collinear.

The **midpoint** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$P_M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Example 3. Find the length of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$.

Definition. A **sphere** is the set of all points that are equidistant from the center.
Problem Find an equation of a sphere of radius R and center $C(a, b, c)$.

Equation of a sphere of radius R and center $C(a, b, c)$ is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

Example 5. Find an equation of a sphere of radius $R = 4$ centered at $C(-1, 2, 4)$.

Example 6. Find radius and center of sphere given by the equation

$$x^2 + y^2 + z^2 + x - 2y + 6z - 2 = 0$$

Example 7. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.