Chapter 9. **Further applications of integration**
Section 9.3 **Arc length**

Let's a curve C is defined by the equations

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

Assume that C is **smooth** ($x'(t)$ and $y'(t)$ are continuous and not simultaneously zero for $a < t < b$)

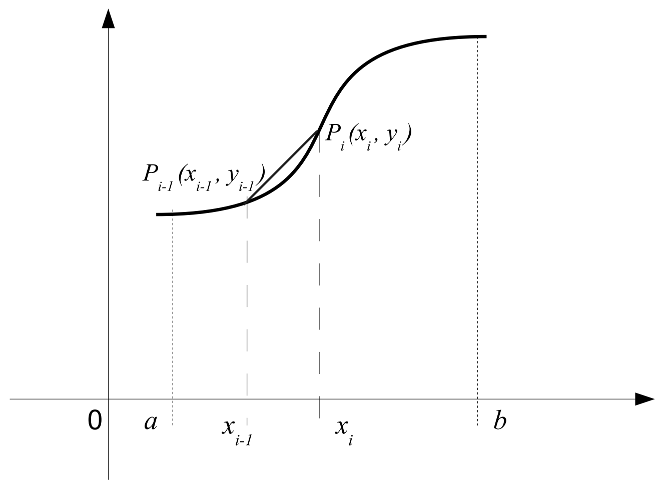
Let P be a partition of $[a, b]$ into n subintervals of equal length Δt.

$$a = t_0 < t_1 < ... < t_n = b, \quad t_i = a + i\Delta t$$

Point $P_i(x(t_i), y(t_i))$ lies on C an the polygon with vertices P_0, P_1,...,P_n approximates C.

We define the **length** of C to be

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} |P_{i-1}\ P_i|$$



Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}\ P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

Since

$$x'(t_i) \approx \frac{\Delta x_i}{\Delta t}, \quad y'(t_i) \approx \frac{\Delta y_i}{\Delta t},$$

then

$$\Delta x_i = x'(t_i)\Delta t, \quad \Delta y_i = y'(t_i)\Delta t$$

Thus,

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2}\Delta t = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt$$

If the curve C is given by the equation

$$y = y(x), \quad a \le x \le b, \quad \text{then} \quad L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx$$

If the curve C is given by the equation

$$x = x(y), \quad c \leq y \leq d, \quad \text{then} \quad L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$$

Example 1. Find the length of the curve

(a) $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 2$

(b) $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $1 \leq x \leq 2$

(c) $x = y^{3/2}$, $0 \leq y \leq 1$