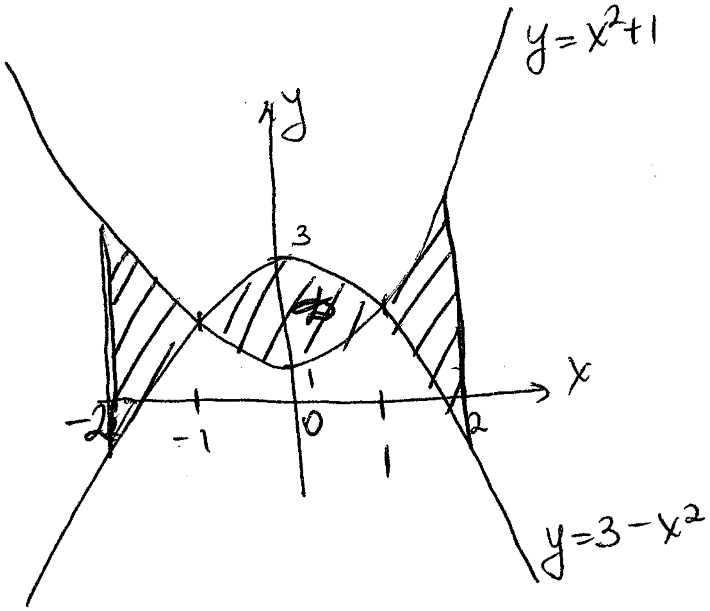
MATH152, 501–506, Spring 2011, Sample problems for Test 1

1. Find the area of the region bounded by $y = x^2 + 1$, $y = 3 - x^2$, $x = -2$, and $x = 2$.



points of intersection:

$x^2+1=3-x^2$

$2x^2=2$

$x^2=1$

$x_1=1, \quad x_2=-1.$

The region $\mathcal{D}$ is symmetric with respect to the y-axis,

so

$$\mathcal{A}(\mathcal{D}) = 2\left(\int_0^1 [(3-x^2)-(x^2+1)]\,dx + \int_1^2 [(x^2+1)-(3-x^2)]\,dx\right)$$

$$=2\left(\int_0^1 (2-2x^2)\,dx + \int_1^2 (2x^2-2)\,dx\right)$$

$$=2\left(\left(2x-\frac{2x^3}{3}\right)\Big|_0^1 + \left(\frac{2x^3}{3}-2x\right)\Big|_1^2\right)$$

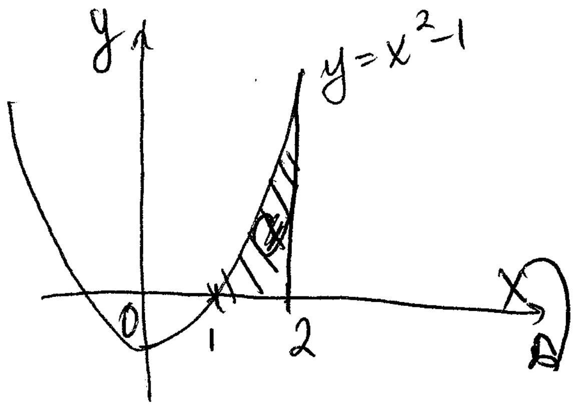
$$=2\left(2-\frac{2}{3}+\frac{2\cdot 8}{3}-4-\frac{2}{3}+2\right)$$

$$=2\left(\cancel{4}-\frac{4}{3}+\frac{16}{3}\cancel{-4}\right)$$

$$=2\,\frac{16-4}{3}$$

$$=\boxed{8}$$

2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 1$, $y = 0$, $x = 1$, $x = 2$ about the x-axis.



disks:

$$V = \pi \int_1^2 [x^2-1]^2 \, dx$$

$$= \pi \int_1^2 (x^4 - 2x^2 + 1) \, dx$$

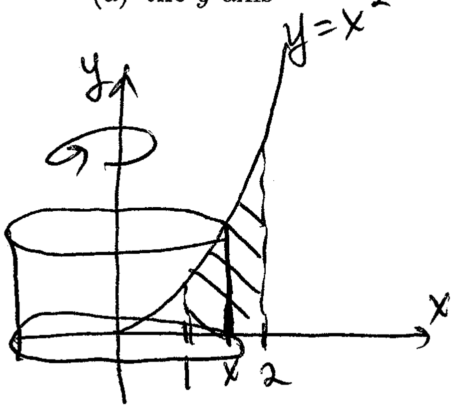
$$= \pi \left(\frac{x^5}{5} - \frac{2x^3}{3} + 1 \right) \Big|_1^2$$

$$= \pi \left(\frac{32}{5} - \frac{16}{3} + 2 - \frac{1}{5} + \frac{2}{3} - 1 \right)$$

$$= \boxed{\frac{38}{15} \pi}$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 0$. $x = 1$, $x = 2$ about

(a) the y-axis



cylindrical shells:

$$V = 2\pi \int_1^2 r\,h\,dx$$

$$r = x,\ h = x^2$$

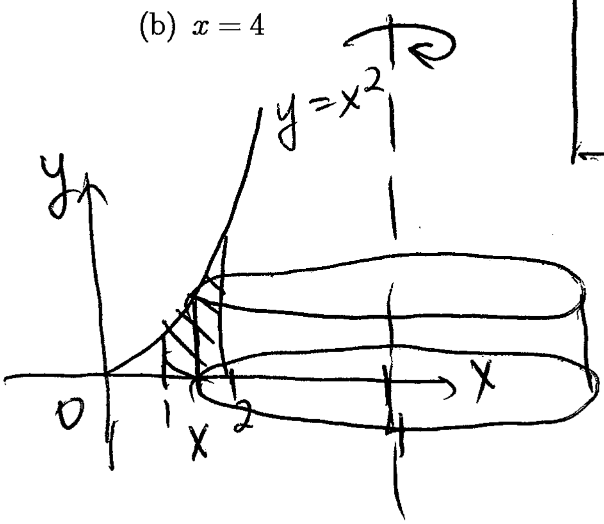
$$V = 2\pi \int_1^2 x\,x^2\,dx$$

$$= 2\pi \int_1^2 x^3\,dx$$

$$= 2\pi\,\frac{x^4}{4}\Big|_1^2$$

$$= 2\pi\left(\frac{16-1}{4}\right) = \boxed{\frac{15\pi}{2}}$$

(b) $x = 4$



cylindrical shells:

$$V = 2\pi \int_1^2 r\,h\,dx$$

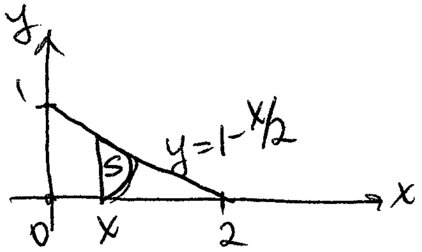
$$r = 4 - x,\ h = x^2$$

$$V = 2\pi \int_1^2 (4-x)\,x^2\,dx = 2\pi \int_1^2 (4x^2 - x^3)\,dx$$

$$= 2\pi \left(\frac{4x^3}{3} - \frac{x^4}{4}\right)\Big|_1^2 = 2\pi\left(\frac{32}{3} - \frac{16}{4} - \frac{4}{3} + \frac{1}{4}\right)$$

$$= 2\pi\left(\frac{28}{3} - \frac{15}{4}\right) = 2\pi\,\frac{67}{12} = \boxed{\frac{67\pi}{6}}$$

4. The base of solid S is the triangular region with vertices (0,0), (2,0), and (0,1). Cross-sections perpendicular to the x-axis are semicircles. Find the volume of S.



$0 \le x \le 2$

$$V=\int_0^2 A(S)\,dx$$

$$A(S)=\frac{1}{2}\pi r^2=\frac{1}{2}\pi\frac{1}{4}\left(1-\frac{x}{2}\right)^2=\frac{1}{8}\pi\left(1-\frac{x}{2}\right)^2$$

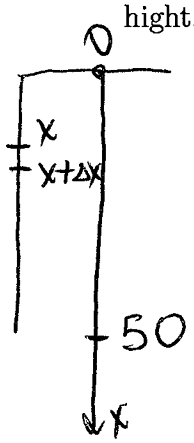
$$V=\frac{\pi}{8}\int_0^2\left(1-\frac{x}{2}\right)^2dx$$

$u=1-\frac{x}{2}$
$du=-\frac{1}{2}dx$
$dx=-2du$
$x=0\to u=1$
$x=2\to u=0$

$$=\frac{\pi}{8}(-2)\int_1^0 u^2\,du$$

$$=-\frac{\pi}{4}\frac{u^3}{3}\Big|_0^1=\boxed{\frac{\pi}{12}}$$

5. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft hight. How much work is done in pulling the rope to the top of the building?



The portion of the rope from x to $x+\Delta x$ weights $0.5\Delta x$ (lb) and must be lifted x (ft)
$0\le x\le 50$

$$W=\int_0^{50}\frac{1}{2}x\,dx=\frac{1}{2}\frac{x^2}{2}\Big|_0^{50}=\frac{1}{4}(2500)$$

$$=\boxed{625\ (\text{ft-lb})}$$

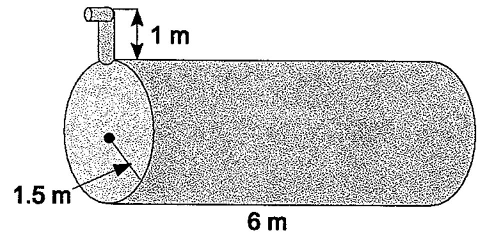
6. A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

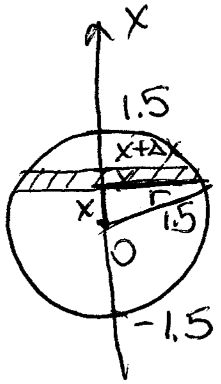
$F=kx$
20 cm $\to$ 0
30 cm $\to$ $30-20=10$ (cm) $=0.1$ (m)
$25=k(0.1)\to k=250\to F=250x$
25 cm $\to$ $25-20=5$ (cm) $=0.05$ (m)

$$W=\int_0^{0.05}250x\,dx=\frac{250}{2}x^2\Big|_0^{0.05}=125(0.0025)$$

$$=\boxed{3.125\ (J)}$$

7. A tank is full of water. Find the work required to pump the water out the outlet.





$-1.5 \le x \le 1.5.$

a slice of water from x to $x+\Delta x$ weights

$(10^3)(9.81)(6)\, r^2 \Delta x$

$r^2 = (1.5)^2 - x^2 = \frac{9}{4} - x^2$

$r = \sqrt{\frac{9}{4} - x^2}$

weight $= (10^3)(9.81)(6)\sqrt{\frac{9}{4}-x^2}\,\Delta x$

the slice must be lifted by

$1.5 - x + 1 = 2.5 - x$ (m).

$$W = \int_{-1.5}^{1.5} (10)^3(9.81)(6)\sqrt{\frac{9}{4}-x^2}\,(2.5-x)\,dx$$

$$= (10)^3(9.81)(6)\left[(2.5)\int_{-1.5}^{1.5}\sqrt{\frac{9}{4}-x^2}\,dx - \int_{-1.5}^{1.5} x\sqrt{\frac{9}{4}-x^2}\,dx\right]$$

odd function

$\shortparallel$

0

area of a semicircle of radius $\frac{3}{2}$

$$= (10)^3(9.81)(6)(2.5)\cdot\frac{1}{2}\pi\frac{9}{4} \approx \boxed{165543{,}75\pi\,(J)}$$

8. Find the average value of $f = \sin^2 x \cos x$ on $[-\pi/2, \pi/4]$.

$$f_{ave} = \frac{1}{\frac{\pi}{4}+\frac{\pi}{2}} \int_{-\pi/2}^{\pi/4} \sin^2 x \cos x \, dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x \, dx \\ x = -\frac{\pi}{2} \to u = -1 \\ x = \pi/4 \to u = \frac{\sqrt{2}}{2} \end{array} \right|$$

$$= \frac{4}{3\pi} \int_{-1}^{\frac{\sqrt{2}}{2}} u^2 du = \frac{4}{3\pi} \frac{u^3}{3} \Big|_{-1}^{\frac{\sqrt{2}}{2}} = \frac{4}{9\pi}\left(\frac{2\sqrt{2}}{8} + 1\right)$$

$$= \boxed{\frac{4}{9\pi}\left(\frac{\sqrt{2}}{4} + 1\right)}$$

9. Evaluate the integral

(a) $\int t^2 \cos(1 - t^3) dt$ $= \left| \begin{array}{l} u = 1 - t^3 \\ du = -3t^2 dt \end{array} \right|$

$$= -\frac{1}{3} \int \cos u \, du = -\frac{1}{3} \sin u + C = -\frac{1}{3} \sin(1 - t^3) + C$$

(b) $\int \frac{x^2}{\sqrt{1-x}} dx$ $= \left| \begin{array}{l} u = 1 - x \\ x = 1 - u \\ dx = -du \end{array} \right| = -\int \frac{(1-u)^2}{\sqrt{u}} du$

$$= -\int \left(\frac{1 - 2u + u^2}{\sqrt{u}}\right) du = -\int \left(\frac{1}{\sqrt{u}} - 2\frac{u}{\sqrt{u}} + \frac{u^2}{\sqrt{u}}\right) du$$

$$= -\int \left(u^{-1/2} - 2u^{1/2} + u^{3/2}\right) du$$

$$= -\left(\frac{u^{-1/2+1}}{-1/2+1} - 2\frac{u^{1/2+1}}{1/2+1} + \frac{u^{3/2+1}}{3/2+1}\right) + C$$

$$= -\left(2u^{1/2} - \frac{2}{3} u^{3/2} + \frac{2}{5} u^{5/2}\right) + C = -2(1-x)^{1/2} + \frac{2}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C$$

(c) $\int_0^1 x^2 e^{-x}\,dx = \left| \begin{array}{ll} f(x)=x^2 & g'(x)=e^{-x} \\ f'(x)=2x & g(x)=-e^{-x} \end{array} \right|$

$$= -x^2 e^{-x}\Big|_0^1 - \int_0^1 2x(-e^{-x})\,dx$$

$$= -e^{-1} + 2\int_0^1 x e^{-x}\,dx \quad \left| \begin{array}{ll} f(x)=x & g'(x)=e^{-x} \\ f'(x)=1 & g(x)=-e^{-x} \end{array} \right|$$

$$= -e^{-1} + 2\left(x(-e^{-x})\Big|_0^1 - \int_0^1 (-e^{-x})\,dx \right)$$

$$= -e^{-1} + 2\left(-e^{-1} - e^{-x}\Big|_0^1 \right)$$

$$= -e^{-1} + 2\left(-e^{-1} - e^{-1} + 1 \right)$$

$$= -e^{-1} + 2\left(-2e^{-1} + 1 \right)$$

$$= \boxed{2 - 3e^{-1}}$$

(d) $\int \sin^3 x \cos^4 x \, dx$ $= \int \sin x \sin^2 x \cos^4 x \, dx =$ $\left| \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \\ \sin^2 x = 1 - \cos^2 x \\ \quad = 1 - u^2 \end{array} \right|$

$$= -\int (1-u^2) u^4 \, du = -\int (u^4 - u^6) \, du$$

$$= -\left(\frac{u^5}{5} - \frac{u^7}{7} \right) + \cancel{4}C$$

$$= \boxed{\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C}$$

(e) $\int_0^{\pi/8} \sin^2(2x) \cos^3(2x) \, dx$ $= \int_0^{\pi/8} \sin^2 2x \cos 2x \cos^2 2x \, dx$

$\left| \begin{array}{l} u = \sin 2x \\ du = 2\cos 2x \, dx \\ x = 0 \to u = \sin 0 = 0 \\ x = \frac{\pi}{8} \to u = \sin \frac{2\pi}{8} = \frac{\sqrt{2}}{2} \\ \cos^2 2x = 1 - \sin^2 2x \\ \quad = 1 - u^2 \end{array} \right|$

$$= \frac{1}{2} \int_0^{\sqrt{2}/2} u^2 (1-u^2) \, du$$

$$= \frac{1}{2} \int_0^{\sqrt{2}/2} (u^2 - u^4) \, du$$

$$= \frac{1}{2} \left(\frac{u^3}{3} - \frac{u^5}{5} \right) \Bigg|_0^{\sqrt{2}/2}$$

$$= \frac{1}{2} \left(\frac{1}{3} \cdot \frac{2\sqrt{2}}{8} - \frac{1}{5} \cdot \frac{4\sqrt{2}}{32} \right)$$

$$= \frac{1}{2} \left(\frac{1}{3} \cdot \frac{\sqrt{2}}{4} - \frac{1}{5} \cdot \frac{\sqrt{2}}{8} \right)$$

$$= \frac{\sqrt{2}}{2 \cdot 4} \left(\frac{1}{3} - \frac{1}{10} \right)$$

$$= \frac{\sqrt{2}}{8} \cdot \frac{7}{10}$$

$$= \boxed{\frac{7\sqrt{2}}{80}}$$

(f) $\int \sin^2 x \cos^4 x \, dx = \int (\sin^2 x \cos^2 x) \cos^2 x \, dx$

$$= \int \frac{1}{4} \sin^2 2x \cos^2 x \, dx$$

$$= \int \frac{1}{4} \sin^2 2x \frac{1+\cos 2x}{2} \, dx$$

$$= \frac{1}{8} \int \sin^2 2x \, dx + \frac{1}{8} \int \sin^2 2x \cos 2x \, dx$$

$u = \sin 2x$
$du = 2\cos 2x \, dx$

$$= \frac{1}{8} \int \frac{1-\cos 4x}{2} \, dx + \frac{1}{8} \cdot \frac{1}{2} \int u^2 \, du$$

$$= \frac{1}{16} \int (1 - \cos 4x) \, dx + \frac{1}{16} \frac{u^3}{3}$$

$$= \frac{1}{16} \left(x - \frac{1}{4} \sin 4x \right) + \frac{1}{48} \sin^3 2x + C$$

$$= \boxed{\frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C}$$

(g) $\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

$$\left| \begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \\ x = 0 \to u = 0 \\ x = \pi/4 \to u = 1 \end{array} \right|$$

$$= \int_0^1 u^4 \, du = \frac{u^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}$$

(h) $\int \tan x \sec^3 x \, dx$

$$= \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array} \right|$$

$$= \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C$$

(i) $\int \sin 3x \cos x \, dx$

$$= \frac{1}{2} \int \left(\sin(3x - x) + \sin(3x + x) \right) dx$$

$$= \frac{1}{2} \int (\sin 2x + \sin 4x) \, dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x \right) + C$$

$$= \boxed{-\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C}$$