MATH 152, 525-530, 534-536, Spring 2013, Solutions for sample problems for Test 3

1. Which of the following series is convergent?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7}+1}$

compare with $\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7}} = \sum_{n=1}^{\infty} n^{2-5/7} = \sum_{n=1}^{\infty} n^{9/7}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{9/7} = \infty$

Divergent by the Test for Divergence ($\lim_{n\to\infty} a_n \neq 0$)

(b) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$ $\qquad 0 \le \cos^2 n \le 1$

$\frac{\cos^2 n}{3^n} \le \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric series, $r = \frac{1}{3} < 1$)

By Comparison Test 1, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$ converges.

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Do the Integral Test.

$f(x) = \frac{1}{x(\ln x)^2}$ is positive on $[2, \infty)$

$x(\ln x)^2$ turns zero at $x=0$, $x=1$.

$\frac{1}{x(\ln x)^2}$ is continuous on $[2,\infty)$.

$f'(x) = -1(x(\ln x))^{-2}\left[(\ln x)^2 - 2x\ln x \cdot \frac{1}{x}\right]$

$= -1\left(\frac{\ln^2 x - 2}{x^2(\ln x)^4}\right) = \frac{2-\ln^2 x}{x^2 \ln^4 x} < 0$

$2 - \ln^2 x < 0$, $\ln^2 x > 2$, $\ln x > \sqrt{2}$

$x > e^{\sqrt{2}} \approx 4.06$

$f(x)$ is decreasing on $[4.06, \infty)$

Can do the Integral Test.

$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \left| \begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \end{array} \right| = \int_{\ln 2}^{\infty} \frac{du}{u^2} = -\frac{1}{u}\Big|_{\ln 2}^{\infty} = 0 - \frac{1}{\ln 2} < \infty$

So since $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \frac{1}{\ln 2}$ convergent,

then, by the Integral Test,

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

2. Approximate the sum of the series $\sum_{n=1}^{\infty} ne^{-n^2}$ by using the sum of first 4 terms. Estimate the error involved in this approximation.

$$\sum_{n=1}^{\infty} ne^{-n^2} \approx e^{-1} + 2e^{-4} + 3e^{-9} + 4e^{-16}$$

Error: $\cancel{|R_4| \le}$ $\int_5^{\infty} xe^{-x^2}dx \le R_4 \le \int_4^{\infty} xe^{-x^2}dx$

$$\int_a^{\infty} xe^{-x^2}dx = \left| \begin{matrix} u=-x^2 \\ du=-2x\,dx \end{matrix} \right| = \lim_{t\to\infty}\left[\int_a^t xe^{-x^2}dx\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}\int_{-a^2}^{-t^2} e^u du\right] = \lim_{t\to\infty} -\frac{1}{2}e^u\Big|_{-a^2}^{-t^2} = -\frac{1}{2}\lim_{t\to\infty} e^{-t^2} + \frac{1}{2}e^{-a^2}$$

$$= \frac{1}{2}e^{-a^2}$$

$$\boxed{\frac{1}{2}e^{-25} \le R_4 \le \frac{1}{2}e^{-16}}$$

3. Approximate the sum of the series $\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n^2}$ by using the sum of first 4 terms. Estimate the error involved in this approximation.

$$\sum_{n=1}^{\infty}(-1)^{n-1} ne^{-n^2} \approx e^{-1} - 2e^{-4} + 3e^{-9} - 4e^{-16}$$

$|R_4| \le b_5$, where $b_n = ne^{-n^2}$

$|R_4| \le 5e^{-25}$

4. Which of the following series is absolutely convergent?

(a) $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

Ratio Test for $a_n = \frac{(-3)^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(-3)}{n+1}\right| = 0 < 1$$

converges absolutely

(b) $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$ - diverges (harmonic series).

$b_n = \frac{1}{n}$ $\quad b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0.$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges by AST, but not absolutely converges.

(c) $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{\sqrt{n-2}}$

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n}{\sqrt{n-2}}\right| = \sum_{n=1}^{\infty}\frac{n}{\sqrt{n-2}}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \lim_{n\to\infty}\frac{n}{\sqrt{n}\sqrt{1-\frac{2}{n}}} = \lim_{n\to\infty}\sqrt{n} = \infty.$$

diverges by the Test for Divergence.

AST: $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \infty.$

diverges by AST.

$\boxed{\text{The series diverges.}}$

(d) $\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{3^{3n}}$

$$\sum_{n=0}^{\infty}\left|(-1)^n \frac{2^{2n}}{3^{3n}}\right| = \sum_{n=0}^{\infty} \frac{2^{2n}}{3^{3n}} = \sum_{n=0}^{\infty}\left(\frac{4}{27}\right)^n$$

converges (geometric series for $r=\frac{4}{27}<1$).

The series converges absolutely.

5. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$.

The radius of converges

$R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$, where $c_n=\frac{2^n}{\sqrt{n+3}}$

$$R=\lim_{n\to\infty}\left|\frac{2^n}{\sqrt{n+3}}\cdot\frac{\sqrt{n+4}}{2^{n+1}}\right| = \frac{1}{2}.$$

The interval of convergence:

$$|x-3|<\frac{1}{2}$$
$$-\frac{1}{2}<x-3<\frac{1}{2}$$
$$+\frac{5}{2}<x<\frac{7}{2}$$

End points: $x=+\frac{5}{2} \rightarrow \sum_{n=1}^{\infty} \frac{2^n\left(+\frac{5}{2}-3\right)^n}{\sqrt{n+3}}$

$$= \sum_{n=1}^{\infty} \cancel{(-1)^n} \frac{2^n\left(-\frac{1}{2}\right)^n}{\sqrt{n+3}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$$ – converges but not absolutely.

$x=\frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{7}{2}-3\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}$ diverges.

interval of convergence: $\boxed{\left[\frac{5}{2}, \frac{7}{2}\right)}$.

$\boxed{R=\frac{1}{2}}$