Chapter 10. **Infinite sequences and series**
Section 10.2 **Series**

An expression of the form

$$a_1 + a_2 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n$$

is called an **infinite series** or **series**.

Consider partial sums:

$$\begin{aligned} &s_1 = a_1, \\ &s_2 = a_1 + a_2, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &s_n = a_1 + a_2 + ... + a_n \end{aligned}$$

Definition Given a series $\sum_{n=0}^{\infty} a_n$, and let $S_n = \sum_{k=1}^{n} a_k$. If the sequence $\{s_n\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} s_n = s$, then the series is called **convergent** and we write

$$\sum_{n=0}^{\infty} a_n = s$$

The number s is called the **sum of the series**. Otherwise, the series is called **divergent**.

The **geometric series**

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r}, & \text{if } |r| < 1 \\ \infty, & \text{if } |r| \geq 1 \end{cases}$$

Example 1. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Example 2. Write the number $0.\overline{307}$ as a ratio of integers.

The **harmonic series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

The p-**series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$

Theorem. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n = 0$, we can not conclude that $\sum_{n=1}^{\infty} a_n$ is convergent.

Test for divergence If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 3. Show that $\sum_{n=1}^{\infty} \arctan n$ is divergent.

Theorem, If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (where c is a constant), $\sum_{n=1}^{\infty} (a_n + b_n)$, $\sum_{n=1}^{\infty} (a_n - b_n)$, and:

(i) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$ (ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

NOTE. A finite number of terms can not affect the convergence of the series.

Example 4. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$.