Chapter 10. **Infinite sequences and series**
Section 10.4 **Other Convergence Tests**

An **alternating** series is a series of the form

$$b_1 - b_2 + b_3 - b_4 + ... = \sum_{n=1}^{\infty} (-1)^{n+1} b_n,$$

where $b_n > 0$ for all n.

The Alternating Series Test If the series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ satisfies

(a) $b_{n+1} \leq b_n$ for all n (b) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

Example 1. Test the series for convergence or divergence.

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2^n}$

(b) $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{6n-5}$

Alternating series estimating theorem If $s = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$ is the sum of alternating series that satisfies the Alternating Series Test, then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Definition A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Example 2. Determine whether the series is absolutely convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

The Ratio Test Given a series $\sum_{n=1}^{\infty} a_n$. Let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

1. If $L < 1$, then the series is absolutely convergent
2. If $L > 1$, then the series is divergent
3. If $L = 1$, then the test is inconclusive.

The Root Test Given a series $\sum_{n=1}^{\infty} a_n$. Let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

1. If $L < 1$, then the series is absolutely convergent
2. If $L > 1$, then the series is divergent
3. If $L = 1$, then the test is inconclusive.

Example 3. Test the series for absolutely convergence, convergence or divergence

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1} \right)^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n!}$

(d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5^n}{n^2}$

(e) $\sum_{n=1}^{\infty} \frac{2^{n-1}}{n^n}$