

Section 10.6. Representations of functions as a power series

Geometric series.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Example 1. Find a power series representation for the function and determine the interval of convergence.

1. $\frac{1}{1+x}$

2. $\frac{1}{1-x^2}$

3. $\frac{1}{1+4x^2}$

4. $\frac{1+x^2}{1-x^2}$

Term-by-term differentiation and integration. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

$$f'(x) = c_1 + 2c_2(x - a) + \dots + nc_n(x - a)^{n-1} + \dots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1}$$

$$\int f(x)dx = C + c_0(x - a) + c_1 \frac{(x - a)^2}{2} + \dots + c_n \frac{(x - a)^{n+1}}{n + 1} + \dots =$$
$$C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

The radii of convergence of these series are R . This does not mean that the interval of convergence remains the same.

Example 2. Find a power series representation for the function and determine the radius of convergence.

1. $\frac{1}{(1 + x)^2}$

2. $\ln(1 + x)$

Example 3. Evaluate an indefinite integral $\int \tan^{-1}(x^2) dx$ as a power series.

Example 4. Use a power series to approximate the integral

$$\int_0^{1/2} \tan^{-1}(x^2) dx$$

to six decimal places.