Table of indefinite integrals

1. $\int a dx = ax + C$, a is a constant,
2. $\int x dx = \frac{x^2}{2} + C$,
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$,
4. $\int \frac{1}{x} dx = \ln|x| + C$,
5. $\int e^x dx = e^x + C$,
6. $\int a^x dx = \frac{a^x}{\ln a} + C$,
7. $\int \sin x dx = -\cos x + C$,
8. $\int \cos x dx = \sin x + C$,
9. $\int \tan x dx = -\ln|\cos x| + C$,
10. $\int \cot x dx = \ln|\sin x| + C$,
11. $\int \sec^2 x dx = \tan x + C$,
12. $\int \csc^2 x dx = -\cot x + C$,
13. $\int \sec x \tan x dx = \sec x + C$,
14. $\int \csc x \cot x = -\csc x + C$,
15. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$,
16. $\int \frac{1}{1+x^2} dx = \arctan x + C$.

Definition of a definite integral

If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition points x_0, x_1,...,x_n, where

$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

Choose points $x_i^* \in [x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. If the limit does exist, then f is called **integrable** on the interval $[a, b]$.

In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the limits of integration; a is the **lower limit** and b is the **upper limit**.

The procedure of calculating an integral is called **integration**.

Properties of the definite integral

1. $\int_a^b c dx = c(b - a)$, where c is a constant.
2. $\int_a^b cf(x)dx = c \int_a^b f(x)dx$, where c is a constant.
3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$.
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
6. If $f(x) \geq 0$ for $a < x < b$, then $\int_a^b f(x)dx \geq 0$.
7. If $f(x) \geq g(x)$ for $a < x < b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

8. If $m \le f(x) \le M$ for $a < x < b$, then $m(b-a) \le \int\limits_a^b f(x)dx \le M(b-a)$.

9. $\left|\int\limits_a^b f(x)dx\right| \le \int\limits_a^b |f(x)|dx$

Section 6.4 The fundamental theorem of calculus.

Suppose f is continuous on $[a,b]$.

1. If $g(x) = \int\limits_a^x f(t)dt$, then $g'(x) = f(x)$.

2. $\int\limits_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$, where F is an antiderivative of f.

Example 1. Find the derivative of the function.

1. $g(x) = \int_\pi^x \frac{1}{1+t^4}dt$

   SOLUTION. By the fundamental theorem of calculus, $g'(x) = \dfrac{1}{1+x^2}$

2. $f(x) = \int_x^4 (2+\sqrt{t})^8 dt$

   SOLUTION. $f(x) = \int_x^4 (2+\sqrt{t})^8 dt = -\int_4^x (2+\sqrt{t})^8 dt$.

   By the fundamental theorem of calculus, $f'(x) = -(2+\sqrt{x})^8$

3. $y(x) = \int_{\tan x}^{17} \sin(t^4)dt$

   SOLUTION. $y(x) = \int_{\tan x}^{17} \sin(t^4)dt = -\int_{17}^{\tan x} \sin(t^4)dt$

   Let $u = \tan x$, then $y(x) = -\int_{17}^{u} \sin(t^4)dt$

   By the fundamental theorem of calculus, $y'(x) = -\sin(u^4)\dfrac{du}{dx} = -\sin(\tan^4 x)\sec^2 x$

Example 2. Evaluate the integral.

1. $\displaystyle\int\limits_2^6 \frac{1+\sqrt{y}}{y^2}dy$

   SOLUTION. $\displaystyle\int\limits_2^6 \frac{1+\sqrt{y}}{y^2}dy = \int\limits_2^6 (1+\sqrt{y})y^{-2}dy = \int\limits_2^6 (y^{-2}+y^{-2}\sqrt{y})dy = \int\limits_2^6 (y^{-2}+y^{-3/2})dy$

   $$= \left(\frac{y^{-1}}{-1} + \frac{y^{-1/2}}{-1/2}\right)\Bigg|_2^6 = \left(-\frac{1}{y} - \frac{2}{\sqrt{y}}\right)\Bigg|_2^6 = -\frac{1}{6} - \frac{2}{\sqrt{6}} + \frac{1}{2} + \frac{2}{\sqrt{2}} = \frac{1}{3} - \frac{2}{\sqrt{6}} + \sqrt{2}$$

2. $\displaystyle\int\limits_0^2 f(x)dx$, where $f(x) = \begin{cases} x^4 & 0 \le x < 1 \\ x^5 & 1 \le x \le 2 \end{cases}$

SOLUTION. $\int\limits_0^2 f(x)dx = \int\limits_0^1 f(x)dx + \int\limits_1^2 f(x)dx = \int\limits_0^1 x^4dx + \int\limits_1^2 x^5dx = \left.\frac{x^5}{5}\right|_0^1 + \left.\frac{x^6}{6}\right|_1^2 =$
$\frac{1}{5} + \frac{2^6}{6} - \frac{1}{6} = \frac{1}{5} + \frac{64}{6} - \frac{1}{6} = \frac{107}{10}$

Example 3. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$.

1. Find the displacement of the particle during the time period $1 \leq t \leq 6$.

   SOLUTION. Displacement$=\left|\int\limits_1^6 v(t)dt\right| = \left|\int\limits_1^6 (t^2 - 2t - 8)dt\right| = \left|\left.\left(\frac{t^3}{3} - \frac{2t^2}{2} - 8t\right)\right|_1^6\right| =$
   $\left|\frac{6^3}{3} - 36 - 48 - \frac{1}{3} + 1 + 8\right| = \frac{10}{3}$

2. Find the distance traveled during this time period.

   SOLUTION. Distance$=\int\limits_1^6 |v(t)|dt$

   $v(t) > 0$ on $(4, 6)$ and $v(t) < 0$ on $1, 4$, therefore
   $\int\limits_1^6 |v(t)|dt = -\int \lim_1^4 (t^2 - 2t - 8)dt + \int \lim_4^6 (t^2 - 2t - 8)dt = -\left.\left(\frac{t^3}{3} - \frac{2t^2}{2} - 8t\right)\right|_1^4 +$
   $\left.\left(\frac{t^3}{3} - \frac{2t^2}{2} - 8t\right)\right|_4^6 = -\frac{4^3}{3} + 4^2 + (4)(8) + \frac{1}{3} - 1 - 8 + \frac{6^3}{3} - 6^2 - (6)(8) - \frac{4^3}{3} + 4^2 + (4)(8) = \frac{98}{3}$