Section 6.5 **The Substitution Rule**

The substitution rule for indefinite integrals If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 1. Evaluate each integral:

1. $\int x^2 e^{x^3} dx$

   SOLUTION. $\int x^2 e^{x^3} dx = \left| \begin{array}{l} u = x^3 \\ du = 3x^2 dx \end{array} \right| = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$

2. $\displaystyle\int \frac{x + \arcsin x}{\sqrt{1-x^2}} dx$

   SOLUTION. $\displaystyle\int \frac{x + \arcsin x}{\sqrt{1-x^2}} dx = \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx$

$$\int \frac{x}{\sqrt{1-x^2}} dx = \left| \begin{array}{l} u = 1 - x^2 \\ du = -2x dx \end{array} \right| = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}(2)u^{1/2} + C = -\sqrt{1-x^2} + C$$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \left| \begin{array}{l} u = \arcsin x \\ du = \dfrac{dx}{\sqrt{1-x^2}} \end{array} \right| = \int u\ du = \frac{u^2}{2} + C = \frac{\arcsin^2 x}{2} + C$$

Therefore, $\displaystyle\int \frac{x + \arcsin x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + \frac{\arcsin^2 x}{2} + C$

The substitution rule for definite integrals If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 2. Evaluate the integral:

1. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

$$\text{SOLUTION. } \int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \left| \begin{array}{l} u = \ln x \\ du = \dfrac{dx}{x} \\ e \to \ln e = 1 \\ e^4 \to \ln(e^4) = 4 \end{array} \right| = \int_1^4 \frac{du}{\sqrt{u}} = 2\ u^{1/2}\Big|_1^4 = 2(4^{1/2} - 1) = 2$$

2. $\int\limits_0^1 \frac{xdx}{\sqrt{1+x^4}}$

SOLUTION. $\int\limits_0^1 \frac{xdx}{\sqrt{1+x^4}} = \left| \begin{array}{l} u = x^2 \\ du = 2x\ dx \\ 0 \to 0^2 = 0 \\ 1 \to 1^2 = 1 \end{array} \right| = \frac{1}{2}\int_0^1 \frac{du}{\sqrt{1+u^2}} = \frac{1}{2}\ln|u+\sqrt{1+u^2}|\Big|_0^1 =$
$\frac{1}{2}\ln(1+\sqrt{2})$

If $F(x)$ is an antiderivative to $f(x)$, then

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$$

Example 3. Evaluate

1. $\int \sin 5x dx$

SOLUTION. $\int \sin 5x dx = \left| \begin{array}{l} u = 5x \\ du = 5dx \end{array} \right| = \frac{1}{5}\int \sin u\ du = -\frac{1}{5}\cos u + C = -\frac{1}{5}\cos 5x + C$

2. $\int \frac{dx}{\sqrt{3x+1}}$

SOLUTION. $\int \frac{dx}{\sqrt{3x+1}} = \left| \begin{array}{l} u = 3x+1 \\ du = 3dx \end{array} \right| = \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}(2)u^{1/2} + C = \frac{2}{3}(3x+1)^{1/2} + C$

Integrals of symmetric functions Suppose f is continuous on $[-a, a]$.
(a) If f is **even**, then $\int\limits_{-a}^{a} f(x)dx = 2\int\limits_0^a f(x)dx$

(b) If f is **odd**, then $\int\limits_{-a}^{a} f(x)dx = 0$

Example 4. Evaluate the integral $\int\limits_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6}dx$.

SOLUTION. Since $\frac{x^2\sin x}{1+x^6}$ is odd, then $\int\limits_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6}dx = 0$