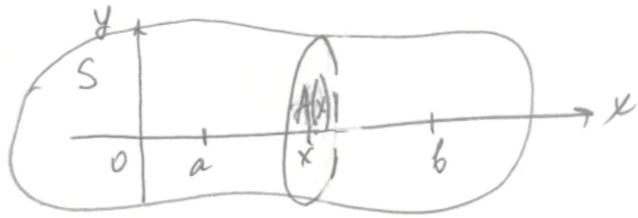
Section 7.2 **Volume**

We start with a simple type of solid called a **cylinder**. A cylinder is bounded by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments perpendicular to the base that join B_1 and B_2. If the area of the base is A and the height of the cylinder is h, then the volume of the cylinder is defined as $V = Ah$.

Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S. Suppose that the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x is $A(x)$, where $a \le x \le b$.



Let's consider a partition P of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$. The planes P_{x_i} will slice S into smaller "slabs". If we choose x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S between $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and height $\Delta x_i = x_i - x_{i-1}$.

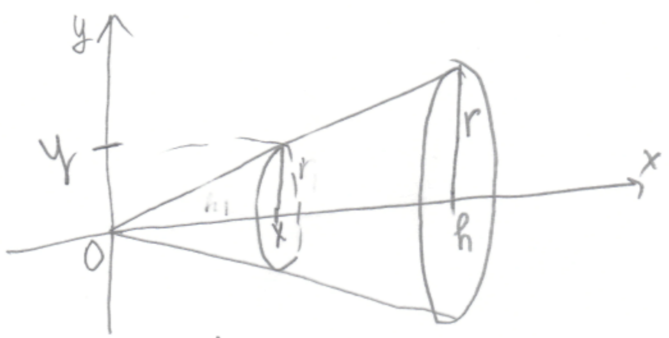
The volume of this cylinder is $A(x_i^*)\Delta x_i$, so the approximation to volume of the ith slab is $V(S_i) \approx A(x_i^*)\Delta x_i$. Thus, the approximation to the volume of S is $V \approx \sum\limits_{i=1}^{n} A(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

Definition of volume Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is an integrable function, then the **volume** of S is

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)dx$$

IMPORTANT. $A(x)$ is the area of a moving cross-sectional obtained by slicing through x perpendicular to the x-axis.

Example 1. Find the volume of a right circular cone with height h and base radius r.



$A = \pi y^2 = \pi \frac{r^2}{h^2} x^2$

$\frac{x}{y} = \frac{h}{r}$

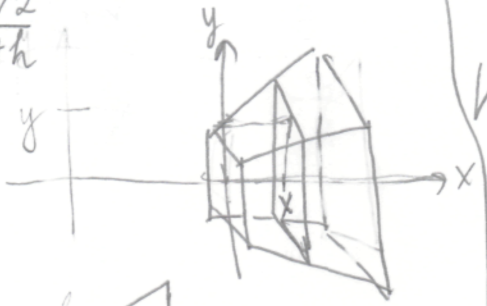
$y = \frac{xr}{h}$

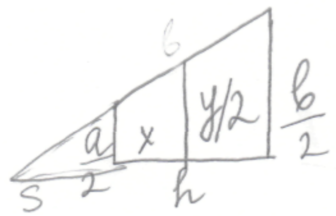
$$V = \int_0^h \pi \frac{r^2}{h^2} x^2 dx = \frac{\pi r^2}{h^2} \frac{x^3}{3}\Big|_0^h = \frac{\pi r^2}{h^2} \frac{h^3}{3} = \boxed{\frac{1}{3}\pi r^2 h}$$

Example 2. Find the volume of a frustum of a pyramid with square base of side b, square top of side a, and height h.

$$\frac{a/2}{s} = \frac{y/2}{s+x} = \frac{b/2}{s+h}$$

$$y = \frac{x}{h}(b-a) + a$$





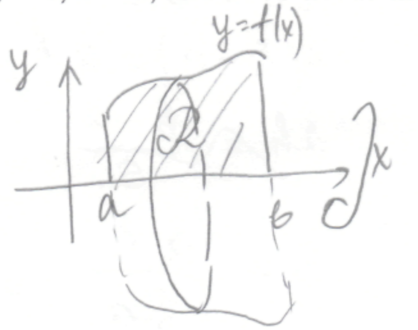
$$A = y^2 = \frac{x^2}{h^2}(b-a)^2 + 2\frac{xa}{h}(b-a) + a^2$$

$$V = \int_0^h A\,dx = \frac{(b-a)^2}{h^2}\frac{x^3}{3}\Big|_0^h + \frac{2a}{h}(b-a)\frac{x^2}{2}\Big|_0^h + a^2 x\Big|_0^h = \frac{(b-a)^2}{3}h + ha(b-a) + a^2 h$$

$$= \frac{b^2 - 2ab + a^2}{3}h + hab - \cancel{ha^2} + \cancel{a^2 h}$$

$$= \frac{h}{3}(b^2 + ab + a^2)$$

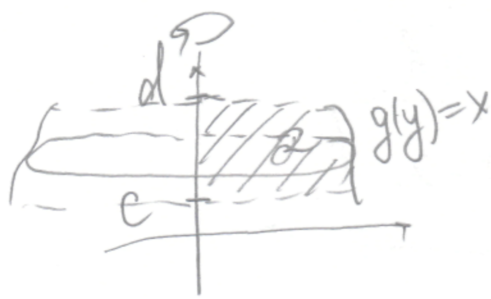
Volume by disks. Let S be the solid obtained by revolving the plane region $\mathcal{R}$ bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the x-axis.



A cross-section through x perpendicular to the x-axis is a circular disc with radius $|y| = |f(x)|$, the cross-sectional area is $A(x) = \pi y^2 = \pi[f(x)]^2$, thus, we have the following **formula for a volume of revolution:**

$$V_X = \pi \int_a^b [f(x)]^2 dx$$

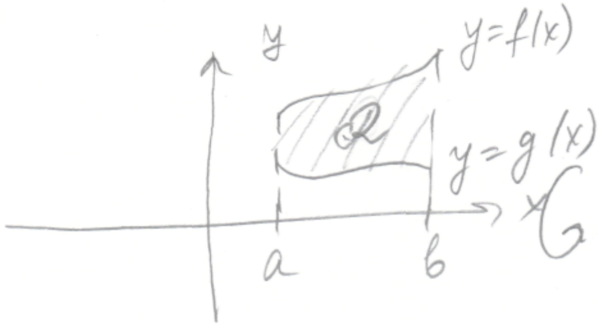
The region bounded by the curves $x = g(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the y-axis.



Then the corresponding volume of revolution is

$$V_Y = \pi \int_c^d [g(y)]^2 dy$$

Volume by washers. Let S be the solid generated when the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ (where $f(x) \geq g(x)$ for all x in $[a, b]$) is rotated about the x-axis.

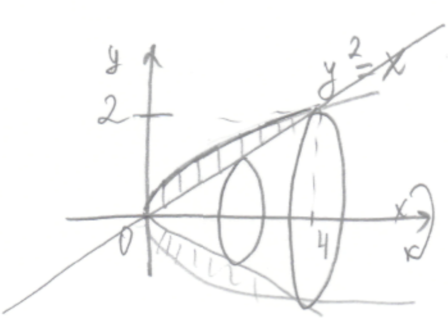


Then the volume of S is

$$V_X = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\} dx$$

Example 3.

1. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$, $x = 2y$ about the x-axis.



$y^2 = 2y$

$y(y-2) = 0$

$y_1 = 0, \quad y_2 = 2.$

$x_1 = 0, \quad x_2 = 4$

$y^2 = x \longrightarrow y = \sqrt{x}$

$x = 2y \longrightarrow y = \frac{x}{2}$

$$V_X = \pi \int_0^4 \left[(\sqrt{x})^2 - \left(\frac{x}{2}\right)^2 \right] dx$$

$$= \pi \int_0^4 \left(x - \frac{x^2}{4}\right) dx$$

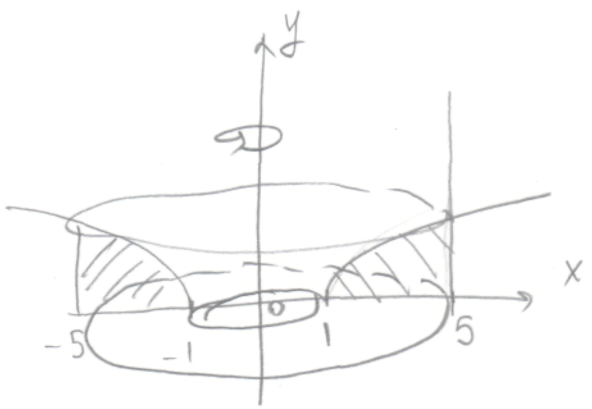
$$= \pi \left(\frac{x^2}{2} - \frac{x^3}{12} \right) \Big|_0^4 = \pi \left(\frac{16}{2} - \frac{64}{12} \right) = \pi \left(8 - \frac{16}{3} \right)$$

$$= \pi \frac{8}{3} = \boxed{\frac{8\pi}{3}}$$

2. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about the y-axis.

$y = \sqrt{5-1} = 2$

$y = \sqrt{x-1}$
$x = y^2 + 1$



$$V_y = \pi \int_0^2 [25 - (y^2+1)^2]\, dy$$

$$= \pi \int_0^2 [25 - y^4 - 1 - 2y^2]\, dy$$

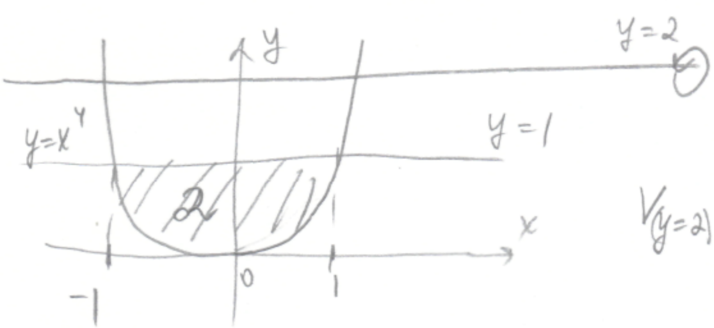
$$= \pi \left[24y - \frac{y^5}{5} - \frac{2y^4}{4}\right]_0^2$$

$$= \pi\left(48 - \frac{32}{5} - 32\right)$$

$$= \pi\left(16 - \frac{32}{5}\right)$$

$$= \boxed{\pi \frac{48}{5}}$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 1$ about the line $y = 2$.



$$V_{(y=2)} = \pi \int_{-1}^{1} [(2-x^4)^2 - 1]\, dx$$

$$= 2\pi \int_0^1 (4 - 4x^4 + x^8 - 1)\, dx$$

$$= 2\pi \left(3x - \frac{4x^5}{5} + \frac{x^9}{9}\right)\Big|_0^1$$

$$= 2\pi\left(3 - \frac{4}{5} + \frac{1}{9}\right)$$

$$= 2\pi \, \frac{135 - 36 + 5}{45}$$

$$= \frac{208\pi}{45}$$