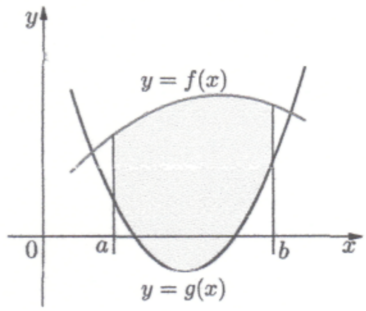
Chapter 7. **Applications of integration**
Section 7.1 **Areas between curves**

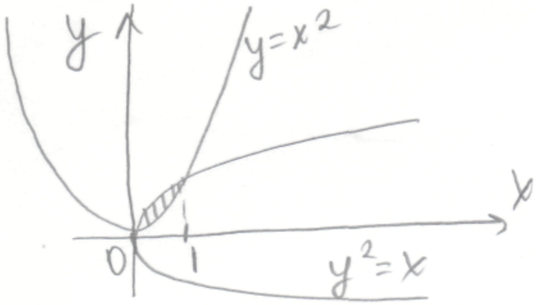
The area of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]dx$$



Example 1. Find the area of the region bounded by

1. $y = x^2$, $y^2 = x$



Points of intersection:

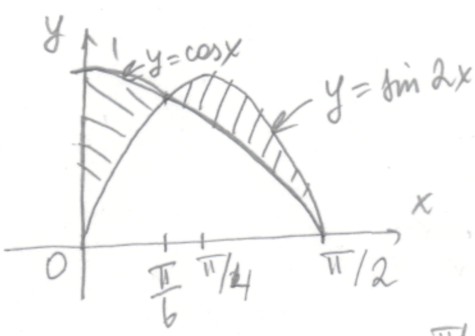
$x^4 = x$, $x^4 - x = 0$, $x(x^3 - 1) = 0$

$x_1 = 0$, $x_2 = 1$

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

2. $y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$



points of intersection:

$\cos x = \sin 2x$

$\cos x = 2\sin x \cos x$

$(2\sin x - 1)\cos x = 0$

$\cos x = 0$ or $2\sin x - 1 = 0$

$x = \frac{\pi}{2}$ $\quad$ $\sin x = \frac{1}{2}$

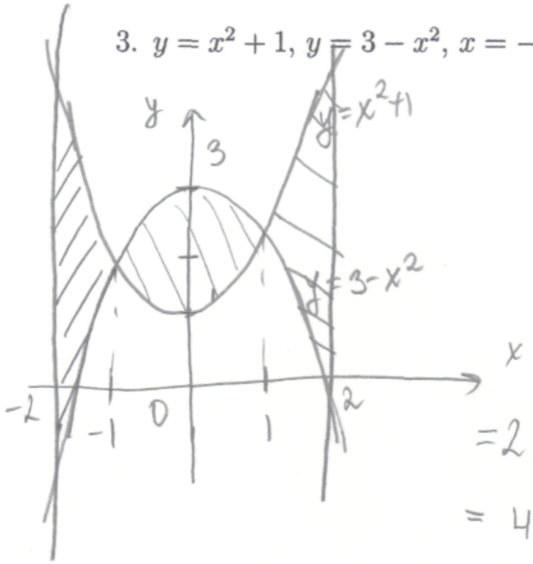
$x = \frac{\pi}{6}$

$$A = \int_0^{\pi/6} (\cos x - \sin 2x)dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)dx = \sin x\Big|_0^{\pi/6} + \frac{1}{2}\cos 2x\Big|_0^{\pi/6} - \frac{1}{2}\cos 2x\Big|_{\pi/6}^{\pi/2}$$

$$- \sin x\Big|_{\pi/6}^{\pi/2} = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} - \frac{1}{2}\cos 0 - \frac{1}{2}\cos\pi + \frac{1}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{2} + \sin\frac{\pi}{6}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\frac{1}{2} - 1 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

3. $y = x^2 + 1$, $y = 3 - x^2$, $x = -2$, $x = 2$



Points of intersection:

$x^2+1 = 3-x^2$

$2x^2 = 2$

$x = \pm 1$

$A = 2A_1 = 2\left(\int_0^1 (3-x^2-x^2-1)dx + \int_1^2 (x^2+1-3+x^2)dx\right)$

$= 2\left(\int_0^1 (2-2x^2)dx + \int_1^2 (2x^2-2)dx\right)$

$= 4\left(\left(x-\frac{x^3}{3}\right)\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 - x\Big|_1^2\right) = 4\left(1-\frac{1}{3}+\frac{8}{3}-\frac{1}{3}-2+1\right)$

$= \boxed{8}$

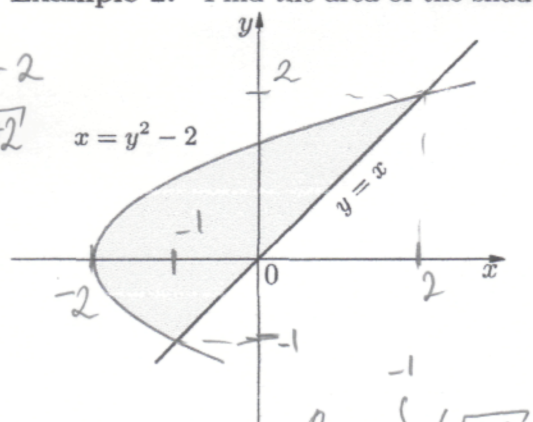
In general case, the area between the curves $y = f(x)$, $y = g(x)$ and between $x = a$ and $x = b$, is

$$A = \int_a^b |f(x) - g(x)|dx$$

Example 2. Find the area of the shaded region.

$x = y^2 - 2$

$y = \pm\sqrt{x+2}$



Points of intersection:

$y^2 - 2 = y$

$y^2 - y - 2 = 0$

$y_1 = -1$, $y_2 = 2$

$x_1 = -1$, $x_2 = 2$.

$A = \int_{-2}^{-1} \left(\sqrt{x+2} - (-\sqrt{x+2})\right)dx + \int_{-1}^{2} \left(\sqrt{x+2} - x\right) dx$

or

$A = \int_{-1}^{2} (y - (y^2-2))dy = \left(\frac{y^2}{2} - \frac{y^3}{3} + 2y\right)\Big|_{-1}^{2} = \frac{4}{2} - \frac{8}{3} + 4 - \left(\frac{1}{2} + \frac{1}{3} - 2\right)$

$= 2 - \frac{8}{3} + 4 - \frac{1}{2} - \frac{1}{3} + 2 = \boxed{\frac{9}{2}}$

If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous functions and $f(y) \geq g(y)$ for all y in $[c, d]$, then its area is

$$A = \int_{c}^{d} [f(y) - g(y)] dy$$