Section 7.3 **Volumes by cylindrical shells**

Lets find the volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h (see Fig.1).

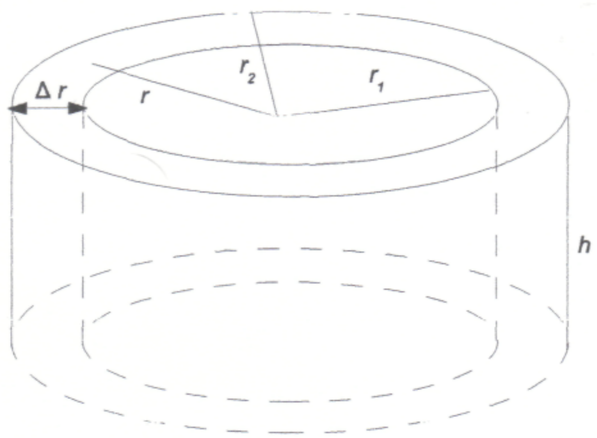


Fig.1

V can be calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

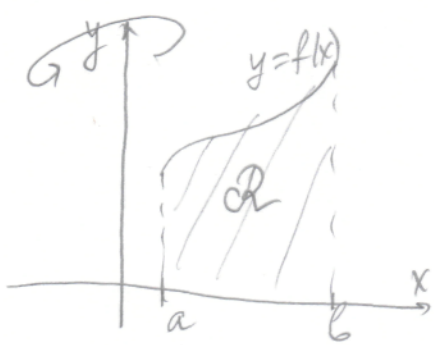
$$V = V_2 - V_1 = \pi h(r_2^2 - r_1^2) = 2\pi h \frac{r_2 + r_1}{2}(r_2 - r_1)$$

Let $\Delta r = r_2 - r_1$, $r = (r_2 + r_1)/2$, then the volume of a cylindrical shell is

$$V = 2\pi r h \Delta r$$

$V =$[circumference][height][thickness]
$V = 2\pi$[average radius][height][thickness]

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$.



Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$ and let x_i^* be the midpoint of $[x_{i-1}, x_i]$, that is $x_i^* = (x_{i-1} + x_i)/2$. If the rectangle with base $[x_{i-1}, x_i]$

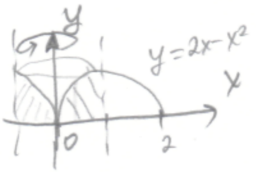
and height $f(x_i^*)$ is rotated about the y-axis, then the result is a cylindrical shell with average raduis x_i^*, height $f(x_i^*)$, and thikness $\Delta x_i = x_i - x_{i-1}$, so its volume is $V_i = 2\pi x_i^* f(x_i^*)\Delta x_i$.

The approximation to the volume V of S is $V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

Thus, the volume of S is

$$V_Y = \lim_{\|P\|\to 0} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i = 2\pi \int_a^b x f(x)dx$$

Example 1. Find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$, $y = 0$, $x = 0$, $x = 1$ about the y-axis.



$$V_y = 2\pi \int_0^1 x(2x - x^2)dx$$

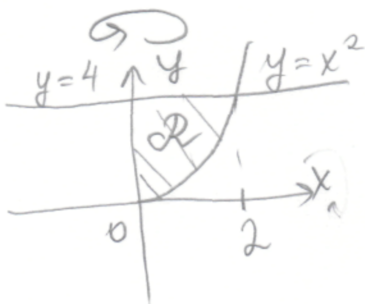
$$= 2\pi \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{2}{3} - \frac{1}{4}\right) = 2\pi \frac{8-3}{12} = \boxed{\frac{5\pi}{6}}$$

The volume of the solid generated by rotating about the y-axis the region between the curves $y = f(x)$ and $y = g(x)$ from a to b [$f(x) \geq g(x)$ and $0 \leq a < b$] is

$$V_Y = 2\pi \int_a^b x[f(x) - g(x)]dx$$

Example 2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 4$, $x = 0$ about the y-axis, $x > 0$.



Point of intersection:

$x^2 = 4$

$x = 2$.

$$V_y = 2\pi \int_0^2 x(4 - x^2)dx$$

$$= 2\pi \int_0^2 (4x - x^3)dx$$

$$= 2\pi\left(2x^2 - \frac{x^4}{4}\right)\Big|_0^2$$

$$= 2\pi\left(8 - \frac{16}{4}\right)$$

$$= \boxed{8\pi}$$

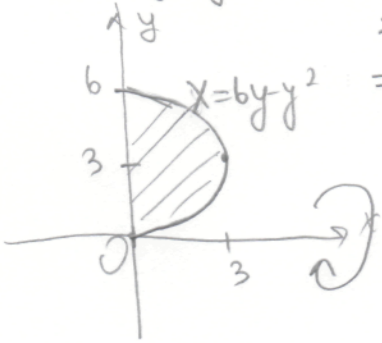
The method of cylindrical shells also allows us to compute volumes of revolution about the x-axis. If we interchange the roles of x and y in the formula for the volume, then the volume of the solid generated by rotating the region bounded by $x = g(y)$, $x = 0$, $y = c$, and $y = d$ about the x-axis, is

$$V_X = 2\pi \int_c^d y g(y) dy$$

Example 3. Find the volume of the solid obtained by rotating the region bounded by $y^2 - 6y + x = 0$, $x = 0$ about the x-axis.

$x = 6y - y^2$

$$x = -(y^2 - 6y) = -(y^2 - 6y + 9) + 9 = -(y-3)^2 + 3^2$$



$$V_x = 2\pi \int_0^6 y(6y - y^2) dy = 2\pi \int_0^6 (6y^2 - y^3)\, dy = 2\pi\left(\frac{6y^3}{3} - \frac{y^4}{4}\right)\Big|_0^6$$

$$= 2\pi\left(2(6)^3 - \frac{6^4}{4}\right) = 2\pi(432 - 324) = \boxed{216\pi}$$

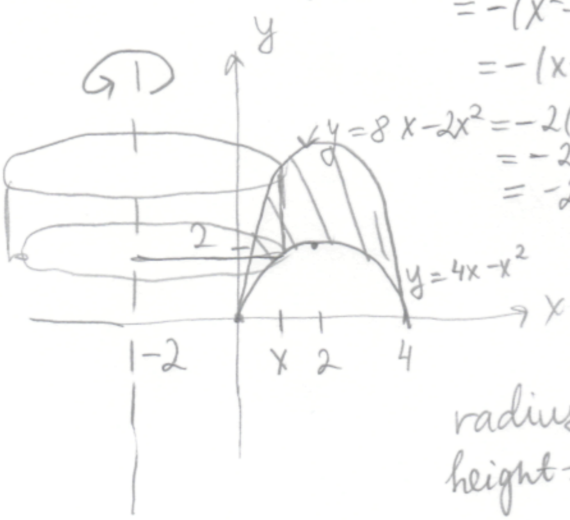
The volume of the solid generated by rotating the region bounded by $x = g_1(y)$, $x = g_2(y)$, $y = c$, and $y = d$, about the x-axis, assuming that $g_2(y) \geq g_1(y)$ for all $c \leq x \leq d$, is

$$V_X = 2\pi \int_c^d y[g_2(y) - g_1(y)] dy$$

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$, $y = 8x - 2x^2$ about $x = -2$.

$$y = 4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = -(x-2)^2 + 2^2$$

$$y = 8x - 2x^2 = -2(x^2 - 4x) = -2(x - 4x + 4) + 8 = -2(x-2)^2 + (2\sqrt{2})^2$$



radius $= x + 2$

height $= 8x - 2x^2 - (4x - x^2) = 4x - x^2$

$$V_{(x=-2)} = 2\pi \int_0^4 (\text{radius})(\text{height}) dx$$

$$= 2\pi \int_0^4 (x+2)(4x - x^2) dx$$

$$= 2\pi \int_0^4 (4x^2 - x^3 + 8x - 2x^2) dx$$

$$= 2\pi \int_0^4 (8x + 2x^2 - x^3) dx$$

$$= 2\pi\left(\frac{8x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_0^4$$

$$= 2\pi\left(4x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^4$$

$$= 2\pi\left(64 + \frac{2}{3}(64) - 64\right) = \boxed{\frac{256\pi}{3}}$$