Section 7.5 **Average value of a function**

Let us try to compute the average value of a function $y = f(x)$, $a \le x \le b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b-a)/n$ and choose points x_i^* in successive subintervals. Then the average of the numbers $f(x_1^*)$, $f(x_2^*)$,...,$f(x_n^*)$, is

$$\frac{f(x_1^*) + f(x_2^*) + ... + f(x_n^*)}{n}$$

Since $n = (b - a)\Delta x$,

$$\frac{f(x_1^*) + f(x_2^*) + ... + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}(f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x)$$

The limiting value as $n \to \infty$ is

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a} \int_a^b f(x)dx$$

We define the **average value of** f on the interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)dx$$

Example 1. Find the average value of $f(x) = \sin^2 x \cos x$ on $[\pi/4, \pi/2]$.

SOLUTION. $f_{\text{ave}} = \dfrac{1}{\pi/2 - \pi/4} \int_{\pi/4}^{\pi/2} \sin^2 x \cos x \, dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x \\ \pi/4 \to \sin(\pi/4) = \dfrac{\sqrt{2}}{2} \\ \pi/2 \to \sin(\pi/2) = 1 \end{array} \right| = \dfrac{4}{\pi} \int_{\sqrt{2}/2}^{1} u^2 du =$

$\dfrac{4}{\pi} \dfrac{u^3}{3}\Big|_{\sqrt{2}/2}^{1} = \dfrac{4}{3\pi}\left(\dfrac{\sqrt{2}}{4} - 1\right)$

Example 2. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

SOLUTION. $3 = f_{\text{ave}} = \dfrac{1}{b} \int_0^b (2 + 6x - 3x^2)dx = \dfrac{1}{b}(2x + 3x^2 - x^3)\Big|_0^b = \dfrac{1}{b}(2b + 3b^2 - b^3) =$
$2 + 3b - b^2$

Let us solve the equation $3 = 2 + 3b - b^2$ for b

$b^2 - 3b + 1 = 0$

$b_{1,2} = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

Mean value theorem for integrals If f continuous on $[a, b]$, then there exist a number c in $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b - a)$$

The geometric interpretation of this theorem for *positive* functions $f(x)$, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as a region under the graph of f from a to b.

Example 3. Find the average value of the function $f(x) = 4 - x^2$ on the interval $[0, 2]$. Find c $(0 \leq c \leq 2)$ such that $f_{ave} = f(c)$.

SOLUTION. $f\text{ave} = \frac{1}{2} \int_0^2 (4 - x^2) dx = \frac{1}{2} \left(4x - \frac{x^3}{3} \right) \Big|_0^2 = \frac{8}{3}$

$f(c) = 4 - c^2 = \frac{8}{3}$

$c^2 = 4 - \frac{8}{3}$

$c^2 = \frac{4}{3}$

$c = \frac{2}{\sqrt{3}}$