Chapter 8. **Techniques of integration**
Section 8.1 **Integration by parts**

The **formula for integration by parts for indefinite integrals** is

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

The **formula for integration by parts for definite integrals** is

$$\int_a^b f(x)g'(x)dx = f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

Example 1. Find the integral.

1. $\int x\cos 3x\ dx$

   SOLUTION. $\int x\cos 3x\ dx = \left| \begin{array}{ll} f(x) = x & g'(x) = \cos 3x \\ f'(x) = 1 & g(x) = \int \cos 3x\ dx = \frac{1}{3}\sin 3x \end{array} \right|$

   $= \dfrac{x}{3}\sin 3x - \dfrac{1}{3}\int \sin 3x\ dx = \dfrac{x}{3}\sin 3x + \dfrac{1}{9}\cos 3x + C$

2. $\int \ln x\ dx$

   SOLUTION. $\int \ln x\ dx \left| \begin{array}{ll} f(x) = \ln x & g'(x) = 1 \\ f'(x) = \frac{1}{x} & g(x) = x \end{array} \right| = x\ln x - \displaystyle\int x\frac{1}{x}dx = x\ln x - x + C$

3. $\int_0^1 (t^2+2t+3)e^t dt$

   SOLUTION. $\int_0^1 (t^2+2t+3)e^t dt = \left| \begin{array}{ll} f(t) = t^2+2t+3 & g'(t) = e^t \\ f'(t) = 2t+2 & g(t) = e^t \end{array} \right|$

   $= (t^2+2t+3)e^t|_0^1 - \int_0^1 (2t+2)e^t dt = (6e-3) - \int_0^1 (2t+2)e^t dt = \left| \begin{array}{ll} f(t) = 2t+2 & g'(t) = e^t \\ f'(t) = 2 & g(t) = e^t \end{array} \right|$

   $= (6e-3) - \left[(2t+2)e^t|_0^1 - \int_0^1 2e^t dt \right] = (6e-3) - (4e-2) + (2e-2) = 4e-3$

4. $\int e^x \cos x\ dx$

   SOLUTION. $\int e^x\cos x\ dx = \left| \begin{array}{ll} f(x) = \cos x & g'(x) = e^x \\ f'(x) = -\sin x & g(x) = e^x \end{array} \right|$

   $= e^x\cos x + \int e^x \sin x dx = \left| \begin{array}{ll} f(x) = \sin x & g'(x) = e^x \\ f'(x) = \cos x & g(x) = e^x \end{array} \right|$

   $= e^x\cos x + e^x \sin x - \int e^x \cos x dx$

   If we denote $\int e^x\cos x dx = I$, then

   $I = e^x\cos x + e^x\sin x - I$

   Solving for I gives

$I = \frac{e^x}{2}(\cos x + \sin x)$

Therefore,

$\int \cos x e^x dx = \frac{e^x}{2}(\cos x + \sin x)$

5. $\int \sin^{-1} x \; dx$

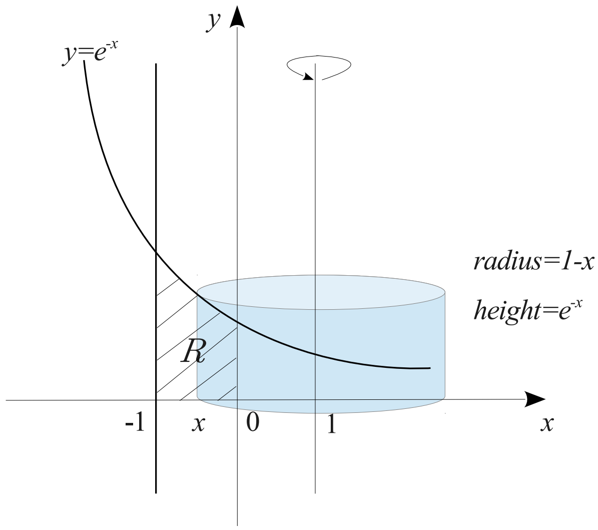
SOLUTION. $\int \sin^{-1} x \; dx = \left| \begin{array}{ll} f(x) = \sin^{-1} x & g'(x) = 1 \\ f'(x) = \frac{1}{\sqrt{1-x^2}} & g(x) = x \end{array} \right|$

$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx = \left| \begin{array}{l} u = 1 - x^2 \\ du = -2x \; dx \end{array} \right| = x \sin^{-1} x + \frac{1}{2} \int \frac{du}{\sqrt{u}}$

$= x \sin^{-1} x + \frac{1}{2} 2u^{1/2} + C = x \sin^{-1} x + \sqrt{1-x^2} + C$

Example 2. Use the methods of cylindrical shells to find the volume generated by rotating the region bounded by $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$ about $x = 1$.

SOLUTION.



$V_{(x=1)} = 2\pi \int\limits_{-1}^{0} (1-x)e^{-x} dx = \left| \begin{array}{ll} f(x) = 1 - x & g'(x) = e^{-x} \\ f'(x) = -1 & g(x) = -e^{-x} \end{array} \right|$

$= 2\pi \left[-(1-x)e^{-x}|_{-1}^{0} - \int\limits_{-1}^{0} (-1)(-e^{-x}) dx \right]$

$= 2\pi(-1 + 2e + e^{-x}|_{-1}^{0}) = 2\pi e$