How to evaluate $\int \sin^m x \cos^n x \; dx$

(a) if $n = 2k+1$ (n is odd), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \; dx = \int \sin^m x (\cos^2 x)^k \cos x \; dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \; dx$$

Then substitute $u = \sin x$

(b) if $m = 2s+1$ (m is odd), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2s+1} x \cos^n x \; dx = \int (\sin^2 x)^s \cos^n x \sin x \; dx = \int (1 - \cos^2 x)^s \cos^n x \sin x \; dx$$

Then substitute $u = \cos x$

Example 1. Evaluate the integral.

1. $\int \cos^3 x \; dx$

   SOLUTION. $\int \cos^3 x \; dx = \int \cos^2 x \cos x \; dx = \int (1 - \sin^2 x) \cos x \; dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x \; dx \end{array} \right|$

   $= \int (1 - u^2) du = u - \dfrac{u^3}{3} + C = \sin x - \dfrac{\sin^3 x}{3} + C$

2. $\int \sin^5 x \cos^4 x \; dx$

   SOLUTION. $\int \sin^5 x \cos^4 x \; dx = \int \sin x \sin^4 x \cos^4 x \; dx = \int \sin x (1 - \cos^2 x)^2 \cos^4 x \; dx$

   $= \left| \begin{array}{l} u = \cos x \\ du = -\sin x \; dx \end{array} \right| = -\int (1 - u^2)^2 \; u^4 \; du = -\int (1 - 2u^2 + u^4) u^4 \; du$

   $= -\int (u^4 - 2u^6 + u^8) du = -\dfrac{u^5}{5} + \dfrac{2u^7}{7} - \dfrac{u^9}{9} + C = -\dfrac{\sin^5 x}{5} + \dfrac{2\sin^7 x}{7} - \dfrac{\sin^9 x}{9} + C$

3. $\displaystyle\int \sin^3 \frac{x}{2} \cos^5 \frac{x}{2} \; dx$

   SOLUTION. $\displaystyle\int \sin^3 \frac{x}{2} \cos^5 \frac{x}{2} \; dx = \int \sin \frac{x}{2} \sin^2 \frac{x}{2} \cos^5 \frac{x}{2} \; dx$

   $\displaystyle= \int \sin \frac{x}{2} \left(1 - \cos^2 \frac{x}{2}\right) \cos^5 \frac{x}{2} \; dx = \left| \begin{array}{l} u = \cos \frac{x}{2} \\ du = -\frac{1}{2} \sin \frac{x}{2} \; dx \end{array} \right| = -2 \int (1 - u^2) u^5 du$

   $\displaystyle= -2 \int (u^5 - u^7) du = -2 \left(\frac{u^6}{6} - \frac{u^8}{8} \right) + C = -\frac{\cos \frac{x}{2}}{3} + \frac{\cos \frac{x}{2}}{4} + C$

(c) if both m and n are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes useful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Example 2. Evaluate each of the following integrals

1. $\int\limits_0^{\pi/2} \sin^2 3x \; dx$

   SOLUTION. $\displaystyle\int_0^{\pi/2} \sin^2 3x \; dx = \int_0^{\pi/2} \frac{1-\cos 6x}{2} dx = \left[\frac{x}{2} - \frac{1}{2}\frac{\sin 6x}{6}\right]_0^{\pi/2}$

   $\displaystyle= \frac{\pi}{4} - \frac{\sin(3\pi)}{12} + \frac{\sin 0}{12} = \frac{\pi}{4}$

2. $\int \cos^4 x \; dx$

   SOLUTION. $\displaystyle\int \cos^4 x \; dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right) dx$

   $\displaystyle= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right) dx$

   $\displaystyle= \frac{1}{4}\left(\frac{3x}{2} + \sin 2x + \frac{\sin 4x}{8}\right) + C = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

3. $\int\limits_0^{\pi/2} \sin^2 x \cos^2 x \; dx$

   SOLUTION. $\displaystyle\int_0^{\pi/2} \sin^2 x \cos^2 x \; dx = \int_0^{\pi/2} \left(\frac{\sin 2x}{2}\right)^2 dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x \; dx$

   $\displaystyle= \frac{1}{4}\int_0^{\pi/2} \frac{1-\cos 4x}{2} \; dx = \left[\frac{x}{8} - \frac{\sin 4x}{32}\right]_0^{\pi/2} = \frac{\pi}{16}$

How to evaluate $\int \tan^m x \sec^n x \; dx$

(a) if the power of secant is even $n = 2k$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \; dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \; dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \; dx$$

Then substitute $u = \tan x$

Example 3.

1. $\int\limits_0^{\pi/4} \tan^4 x\ \sec^2 x\ dx$

   SOLUTION. $\int\limits_0^{\pi/4} \tan^4 x\ \sec^2 x\ dx = \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x\ dx \\ 0 \to \tan 0 = 0 \\ \pi/4 \to \tan(\pi/4) = 1 \end{array} \right| = \int\limits_0^1 u^4 du = \left.\frac{u^5}{5}\right|_0^1 = \frac{1}{5}$

2. $\int \tan^2 x\ dx$

   SOLUTION. $\int \tan^2 x\ dx = \int (\sec^2 x - 1)dx = \tan x - x + C$

3. $\int \tan^4 x\ dx$

   SOLUTION. $\int \tan^4 x\ dx = \int (\sec^2 x - 1)^2 dx = \int (\sec^4 x - 2\sec^2 x + 1)dx$

   $= \int \sec^4 x\ dx - 2\tan x + x = \int \sec^2 x \sec^2 x\ dx - 2\tan x + x$

   $= \int (\tan^2 x + 1)\sec^2 x\ dx - 2\tan x + x = \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x\ dx \end{array} \right| = \int (u^2+1)du - 2\tan x + x$

   $= \frac{u^3}{3} + u - 2\tan x + x + C = \frac{\tan^3 x}{3} + \tan x - 2\tan x + x + C = \frac{\tan^3 x}{3} - \tan x + x + C$

4. $\int \tan^3 x\ dx$

   SOLUTION. $\int \tan^3 x\ dx = \int \tan x \tan^2 x\ dx = \int \tan x(\sec^2 x - 1)dx = \int \tan x \sec^2 x\ dx - \int \tan x\ dx$

   $\int \tan x \sec^2 x\ dx = \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x\ dx \end{array} \right| = \int u\ du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$

   $\displaystyle\int \tan x\ dx = \int \frac{\sin x}{\cos x} dx = \left| \begin{array}{l} v = \cos x \\ dv = -\sin x\ dx \end{array} \right| = -\int \frac{dv}{v} = -\ln|v| + C = -\ln|\cos x| + C$

   Therefore

   $\int \tan^3 x\ dx = \frac{\tan^2 x}{2} - \ln|\cos x| + C$

(b) if the power of tangent is odd ($m = 2s+1$), save a factor of $\tan x \sec x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2s+1} x \sec^n x\ dx = \int (\tan^2 x)^s \sec^m x \tan x \sec x\ dx = \int (\sec^2 x - 1)^s \sec^m x \tan x \sec x\ dx$$

Then substitute $u = \sec x$

Example 4.

1. $\int \tan^3 x\ \sec^3 x\ dx$

   SOLUTION. $\int \tan^3 x\ \sec^3 x\ dx = \int (\tan x \sec x) \tan^2 x \sec^3 x\ dx$

   $= \int (\tan x \sec x)(\sec^2 x - 1)\sec^3 x\ dx \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x\ dx \end{array} \right| = \int (u^2 - 1)u^3\ du$

   $= \int (u^5 - u^3)du = \frac{u^6}{6} - \frac{u^4}{4} + C = \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C$

2. $\int \sec x\ dx$

SOLUTION. $\displaystyle\int \sec x\ dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \left| \begin{array}{l} u = \sec x + \tan x \\ du = (\sec^2 x + \tan x \sec x)dx \end{array} \right|$

$\displaystyle = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$

3. $\int \sec^3 x\ dx$

SOLUTION. $\int \sec^3 x\ dx = \int \sec x \sec^2 x\ dx = \left| \begin{array}{ll} f(x) = \sec x & g'(x) = \sec^2 x \\ f'(x) = \sec x \tan x & g(x) = \tan x \end{array} \right|$

$= \sec x \tan x - \int \sec x \tan^2 x\ dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$

$= \sec x \tan x - \int(\sec^3 x - \sec x)dx = \sec x \tan x + \ln|\sec x + \tan x| - \int(\sec^3 x - \sec x)dx$

Let $I = \int \sec^3 x\ dx$, then

$I = \sec x \tan x + \ln|\sec x + \tan x| - I$

Solving for I gives

$I = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|)$

Therefore

$\int \sec^3 x\ dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|)$

Integrals of the form

$$\int \cot^m x \csc^n x\ dx$$

can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$.

Example 5. Find

1. $\int \cot^4 x \csc^4 x\ dx$

SOLUTION. $\int \cot^4 x \csc^4 x\ dx = \int \cot^4 x \csc^2 x \csc^2 x\ dx = \int \cot^4 x(1 + \cot^2 x)\csc^2 x\ dx$

$\left| \begin{array}{l} u = \cot x \\ du = -\csc^2 x\ dx \end{array} \right| = -\int u^4(1 + u^2)\ du = -\int(u^4 + u^6)du = -\frac{u^5}{5} - \frac{u^7}{7} + C$

$\displaystyle = -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + C$

2. $\int \cot^3 x \csc^2 x\ dx$

SOLUTION. $\int \cot^3 x \csc^2 x\ dx = \left| \begin{array}{l} u = \cot x \\ du = -\csc^2 x\ dx \end{array} \right| = -\int u^3\ du = -\frac{u^4}{4} + C$

$\displaystyle = \frac{\cot^4 x}{4} + C$

To evaluate the integrals

$$\text{(a)} \int \sin mx\ \cos nx\ dx$$

$$\text{(b)} \int \sin mx\ \sin nx\ dx$$

$$\text{(c)} \int \cos mx \ \cos nx \ dx$$

use the corresponding identity:

$$\text{(a)} \ \sin\alpha \ \cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$

$$\text{(b)} \ \sin\alpha \ \sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\text{(c)} \ \cos\alpha \ \cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

Example 6.

1. $\int \sin 5x \ \sin 2x \ dx$

   SOLUTION. $\displaystyle\int \sin 5x \ \sin 2x \ dx = \frac{1}{2}\int (\cos(5x-2x)-\cos(5x+2x))\,dx$

   $\displaystyle = \frac{1}{2}\int (\cos 3x - \cos 7x)dx = \frac{1}{2}\left(\frac{\sin 3x}{3} - \frac{\sin 7x}{7}\right) + C$

2. $\int \sin 3x \ \cos x \ dx$

   SOLUTION. $\displaystyle\int \sin 3x \ \cos x \ dx = \frac{1}{2}\int (\sin(3x-x)+\sin(3x+x))\,dx$

   $\displaystyle = \frac{1}{2}\int (\sin 2x + \sin 4x)\,dx = -\frac{1}{2}\left(\frac{\cos 2x}{2} + \frac{\cos 4x}{4}\right) + C$

3. $\int \cos 3x \ \cos 4x \ dx$

   SOLUTION. $\displaystyle\int \cos 3x \ \cos 4x \ dx = \frac{1}{2}\int (\cos(3x-4x)+\cos(3x+4x))\,dx$

   $\displaystyle = \frac{1}{2}\int (\cos x + \cos 7x)\,dx = \frac{1}{2}\left(\sin x + \frac{\sin 7x}{7}\right) + C$