Chapter 8. **Techniques of integration**
Section 8.3 **Trigonometric substitution**

Assume that g is one-to-one function (g^{-1} exists). Then

$$\int f(x)dx = \int f(g(t))g'(t)dt$$

This kind of substitution is called *inverse substitution*.

Table of trigonometric substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin t, -\pi/2 \le t \le \pi/2$	$1 - \sin^2 t = \cos^2 t$
$\sqrt{a^2 + x^2}$	$x = a\tan t, -\pi/2 < t < \pi/2$	$1 + \tan^2 t = \sec^2 t$
$\sqrt{x^2 - a^2}$	$x = a\sec t, 0 \le t \le \pi/2$ or $\pi \le t \le 3\pi/2$	$\sec^2 t - 1 = \tan^2 t$

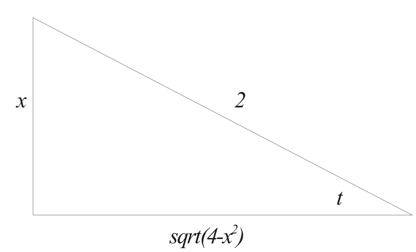
Example Find

(a) $\displaystyle\int x\sqrt{4 - x^2}dx$

SOLUTION. $$\int x\sqrt{4-x^2}dx = \left| \begin{array}{l} x = 2\sin t \\ dx = 2\cos t\ dt \\ \sqrt{4-x^2} = \sqrt{4 - 4\sin^2 t} \\ = \sqrt{4(1-\sin^2 t)} = \sqrt{4\cos^2 t} = 2\cos t \end{array} \right|$$

$$= \int 2\sin t\ 2\cos t\ 2\cos t\ dt = 8\int \sin t\ \cos^2 t\ dt \left| \begin{array}{l} u = \cos t \\ du = -\sin t\ dt \end{array} \right| = -8\int u^2\ du = -\frac{8u^3}{3} + C$$

$$= -\frac{8\cos^3 t}{3} + C$$



sin t=x/2
cos t=sqrt(4-x2)/2

Therefore

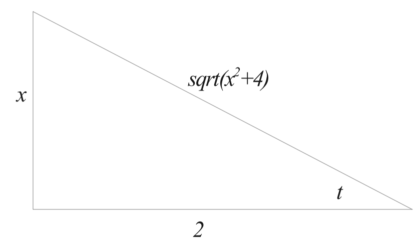
$$\int x\sqrt{4-x^2}dx = -\frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C = -\frac{1}{3}(4-x^2)^{3/2} + C$$

(b) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}dx$

SOLUTION. $$\int \frac{x^3}{\sqrt{x^2+4}}dx = \left| \begin{array}{l} x = 2\tan t \\ dx = 2\sec^2 t\ dt \\ \sqrt{x^2+4} = \sqrt{4\tan^2 t + 4} \\ = \sqrt{4(\tan^2 t + 1)} = \sqrt{4\sec^2 t} = 2\sec t \end{array} \right|$$

$$= \int \frac{8\tan^3 t}{2\sec t} 2\sec^2 t\ dt = 8\int \tan^3 t\ \sec t\ dt = \left| \begin{array}{l} u = \sec t \\ du = \sec t\ \tan t\ dt \\ \tan^2 t = \sec^2 t - 1 = u^2 - 1 \end{array} \right|$$

$$= 8\int (u^2-1)du = 8\left(\frac{u^3}{3} - u\right) + C = 8\left(\frac{\sec^3 t}{3} - \sec t\right) + C$$



tan t=x/2
cos t=2/(sqrt(x²+4))
sec t=sqrt(x²+4)/2

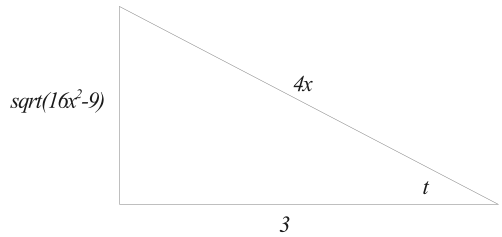
Therefore
$$\int \frac{x^3}{\sqrt{x^2+4}}dx = 8\left[\frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - \frac{\sqrt{x^2+4}}{2}\right] + C$$

(c) $\displaystyle\int \frac{dx}{x^2\sqrt{16x^2-9}}$

SOLUTION. $$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{dx}{x^2\sqrt{16\left(x^2-\frac{9}{16}\right)}} = \int \frac{dx}{4x^2\sqrt{x^2-\frac{9}{16}}}$$

$$= \left| \begin{array}{l} x = \frac{3}{4}\sec t \\ dx = \frac{3}{4}\sec t\ \tan t\ dt \\ \sqrt{x^2-\frac{9}{16}} = \sqrt{\frac{9}{16}\sec^2 t - \frac{9}{16}} \\ = \sqrt{\frac{9}{16}(\sec^2 t - 1)} = \sqrt{\frac{9}{16}\tan^2 t} \\ = \frac{3}{4}\tan t \end{array} \right| = \int \frac{\frac{3}{4}\sec t \tan t\ dt}{4\frac{9}{16}\sec^2 t\frac{3}{4}\tan t} =$$

$$= \frac{4}{9}\int \frac{1}{\sec t}\ dt = \frac{4}{9}\int \cos t\ dt = \frac{4}{9}\sin t + C$$



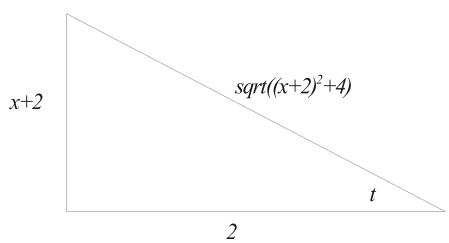
sec t=(4x)/3
cos t=3/(4x)
sin t=sqrt(16x²-9)/(4x)

Therefore
$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \frac{4}{9}\frac{\sqrt{16x^2-9}}{4x} + C = \frac{\sqrt{16x^2-9}}{9x} + C$$

(d) $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+8}}$

SOLUTION. $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{dx}{\sqrt{(x+2)^2+4}} = \left| \begin{array}{l} x+2 = 2\tan t \\ dx = 2\sec^2 t\ dt \\ \sqrt{(x+2)^2+4} = \sqrt{4\tan^2 t+4} \\ = \sqrt{4(\tan^2 t+1)} = \sqrt{4\sec^2 t} = 2\sec t \end{array} \right|$

$$= \int \frac{2\sec^2 t\ dt}{2\sec t} = \int \sec t\ dt = \ln|\sec t + \tan t| + C$$



tan t=(x+2)/2
cos t=2/sqrt((x+2)²+4)
sec t=sqrt((x+2)²+4)/2

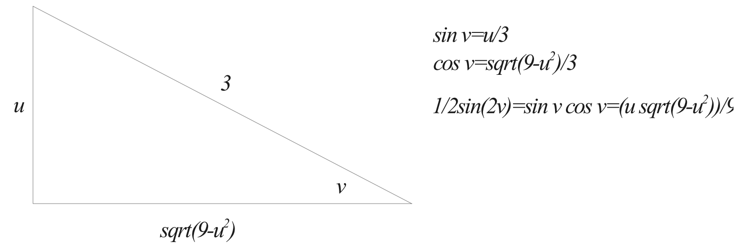
Therefore
$$\int \frac{dx}{\sqrt{x^2+4x+8}} = \ln\left|\frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2}\right| + C$$

(e) $\int e^t\sqrt{9-e^{2t}}\ dt$

SOLUTION. $\int e^t\sqrt{9-e^{2t}}\ dt = \left| \begin{array}{l} u = e^t \\ du = e^t dt \end{array} \right| = \int \sqrt{9-u^2}\ du = \left| \begin{array}{l} u = 3\sin v \\ du = 3\cos v\ dv \\ \sqrt{9-u^2} = 3\cos v \end{array} \right|$

$$= \int 3\cos v\ 3\cos v\ dv = 9\int \cos^2 v\ dv = \frac{9}{2}\int (1+\cos 2v)dv = \frac{9}{2}\left(v + \frac{1}{2}\sin 2v\right) + C$$

$\sin v = \dfrac{u}{3}$, therefore $v = \sin^{-1}\left(\dfrac{u}{3}\right)$.



Therefore

$$\int e^t\sqrt{9-e^{2t}}\ dt = \frac{9}{2}\left(\sin^{-1}\left(\frac{u}{3}\right) + \frac{u\sqrt{9-u^2}}{9}\right) + C = \frac{9}{2}\left(\sin^{-1}\left(\frac{e^t}{3}\right) + \frac{e^t\sqrt{9-e^{2t}}}{9}\right) + C$$