

## Section 9.4 Area of a surface of revolution

A surface of revolution is formed when a curve is rotated about a line.

Let's start with some simple surfaces.

The lateral surface area of a circular cylinder with base radius  $r$  and height  $h$  is

$$A = 2\pi r h$$

The lateral surface area of a circular cone with base radius  $r$  and slant height  $l$  is

$$A = \pi r l$$

The lateral surface area of a band (frustum of a cone) with slant height  $l$ , upper radius  $r_1$  and lower radius  $r_2$  is

$$A = 2\pi r l, \text{ here } r = \frac{r_1 + r_2}{2}$$

Now we consider the surface which is obtained by rotating the curve  $y = f(x)$ ,  $a \leq x \leq b$  about the  $x$ -axis,  $f(x) > 0$  for all  $x$  in  $[a, b]$  and  $f'(x)$  is continuous.

We take a partition  $P$  of  $[a, b]$  by points  $a = x_0 < x_1 < \dots < x_n = b$ , and let  $y_i = f(x_i)$ , so that the point  $P_i(x_i, y_i)$  lies on the curve. The part of the surface between  $x_{i-1}$  and  $x_i$  is approximated by taking the line segment  $P_{i-1}P_i$  and rotating it about the  $x$ -axis. The result is a band with slant height  $|P_{i-1}P_i|$  and average radius  $r = \frac{1}{2}(y_{i-1} + y_i)$ , its surface area is

$$S_i = 2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

We know that

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + [f'(x_i^*)]^2} \Delta x_i$$

where  $x_i^* \in [x_{i-1}, x_i]$ . Since  $\Delta x_i$  is small, we have  $y_i = f(x_i) \approx f(x_i^*)$  and also  $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$  since  $f$  is continuous.

$$S_i \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \Delta x_i$$

Thus, the area of the complete surface is

$$S_X = 2\pi \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \Delta x_i =$$

$$2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx = 2\pi \int_a^b f(x) \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx$$

If the curve is described as  $x = g(y)$ ,  $c \leq y \leq d$ , then the formula for the surface area is

$$S_X = 2\pi \int_c^d y \sqrt{1 + [g'(y)]^2} dy = 2\pi \int_c^d y \sqrt{1 + \left[\frac{dx}{dy}\right]^2} dy$$

Let's a curve  $C$  is defined by the equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

The area of the surface generated by rotating  $C$  about  $x$ -axis is

$$S_X = 2\pi \int_a^b y(t) \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt$$

For rotation about the  $y$ -axis, the surface area formulas are:

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$$S_Y = 2\pi \int_a^b x \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx$$

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$$S_Y = 2\pi \int_a^b x(t) \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt$$

**Example 1.** Find the area of the surface obtained by rotating the curve about  $x$ -axis

(a)  $y = \sqrt{x}$ ,  $4 \leq x \leq 9$

(b)  $y^2 = 4x + 4$ ,  $0 \leq x \leq 8$

(c)  $x(t) = a \cos^3 t$ ,  $y(t) = a \sin^3 t$ ,  $0 \leq t \leq \pi/2$ ,  $a$  is a constant.

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if the curve is given as  $y = f(x)$ ,  $a \leq x \leq b$ , then the formula for the surface area is

$$S_Y = 2\pi \int_a^b x \sqrt{1 + \left[ \frac{df}{dx} \right]^2} dx$$

if the curve is described as  $x = g(y)$ ,  $c \leq y \leq d$ , then the formula for the surface area is

$$S_Y = 2\pi \int_c^d g(y) \sqrt{1 + \left[ \frac{dg}{dy} \right]^2} dy$$

if the is defined by the equations  $x = x(t), y = y(t)$ ,  $a \leq t \leq b$ , then the area of the surface is

$$S_Y = \int_a^b x(t) \sqrt{\left[ \frac{dx}{dt} \right]^2 + \left[ \frac{dy}{dt} \right]^2} dt$$

**Example 2.** Find the area of the surface obtained by rotating the curve about  $y$ -axis

(a)  $x = \sqrt{2y - y^2}$ ,  $0 \leq y \leq 1$

(b)  $y = 1 - x^2$ ,  $0 \leq x \leq 1$

(c)  $x = e^t - t, y = 4e^{t/2}, 0 \leq t \leq 1$