

#66. Find the Maclaurin series for $f(x) = \frac{x^2}{(1+9x)^3}$.

$$\text{We start with } \frac{1}{1+9x} = \sum_{n=0}^{\infty} (-9x)^n = \sum_{n=0}^{\infty} (-1)^n 9^n x^n$$

$$\begin{aligned} \text{then, } \frac{1}{(1+9x)^2} &= -\frac{1}{9} \left(\frac{1}{1+9x} \right)' = -\frac{1}{9} \left(\sum_{n=0}^{\infty} (-1)^n 9^n x^n \right)' = -\frac{1}{9} \sum_{n=1}^{\infty} (-1)^n 9^n n x^{n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} 9^{n-1} n x^{n-1} \end{aligned}$$

$$\begin{aligned} \frac{1}{(1+9x)^3} &= -\frac{1}{18} \left(\frac{1}{(1+9x)^2} \right)' = -\frac{1}{18} \left(\sum_{n=1}^{\infty} (-1)^{n+1} 9^{n-1} n x^{n-1} \right)' \\ &= -\frac{1}{18} \sum_{n=2}^{\infty} (-1)^{n+1} 9^{n-1} n(n-1) x^{n-2} \\ &= \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n 9^{n-2} n(n-1) x^{n-2} \end{aligned}$$

$$\begin{aligned} \frac{x^2}{(1+9x)^3} &= x^2 \cdot \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n 9^{n-2} n(n-1) x^{n-2} \\ &= \boxed{\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n 9^{n-2} n(n-1) x^n} \end{aligned}$$